14.452 Recitation 4: OLG, Romer

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Recitation Plan: Solve the canonical OLG model (Q3 on Problem Set 3) and the Romer (1990) "lab equipment" model with knowledge spillovers

1 OLG: Problem Set 3 Question 3

Consider the two-period canonical overlapping generations model with log preferences

$$\log(c_1(t)) + \beta \log(c_2(t+1))$$

for each individual. Suppose that there is population growth at the rate n. Individuals work only when they are young, and supply one unit of labor inelastically. The production technology is given by

$$Y(t) = A(t) K(t)^{\alpha} L(t)^{1-\alpha},$$

where $A(t+1) = (1+g)A(t)$, with $A(0) > 0$ and $g > 0$.

Part 1. Define a competitive equilibrium and the steady-state equilibrium.

Solution: A *competitive equilibrium* is an allocation $[c_1(t), c_2(t), K(t)]_{t\geq 0}$ and prices $[R(t), w(t)]_{t\geq 0}$ such that

(i) the consumption values $c_1(t)$ and $c_2(t+1)$ are determined by the solution to generation t's optimization problem, taking $w(t)$ and $R(t+1)$ as given:

$$\max_{c_1(t), c_2(t+1)} \log(c_1(t)) + \beta \log(c_2(t+1)) \quad \text{s.t.} \quad c_1(t) + \frac{c_2(t+1)}{R(t+1)} \leq w(t),$$

and $c_2(0) = R(0)K(0)$;

(ii) prices $R(t)$ and $w(t)$ are given by the representative firm's optimality conditions

$$R(t) = F_K(K(t), L(t)) \quad \text{and} \quad w(t) = F_L(K(t), L(t));$$

(iii) capital accumulates according to $K(t+1) = S(t)$, where $S(t) = L(t)(w(t) - c_1(t))$, with $K(0) > 0$ given.

A *steady-state* equilibrium is a competitive equilibrium in which output $Y(t)$, capital $K(t)$, and total consumption $C(t) = c_1(t)L(t) + c_2(t)L(t-1)$ all grow at constant rates.

Part 2. Can you apply the First Welfare Theorem to this competitive equilibrium?

Solution: Not necessarily – see the discussion below.

Part 3. Characterize the steady-state equilibrium and show that it is globally asymptotically stable.

Solution: Begin by writing generation t's problem as an optimization problem over savings $s(t) = w(t) - c_1(t)$:

$$\max_{s(t)\in[0,w(t)]} \log\left(w(t) - s(t)\right) + \beta \log\left(R(t+1)s(t)\right).$$

The solution implies that the household saves a constant fraction of its income (wealth), regardless of the interest rate $R(t+1)$:

$$s(t) = \frac{\beta}{1+\beta} w(t).$$

This owes to the assumption of log preferences, which are equivalently Cobb-Douglas preferences over consumption when young and when old (take the exponential of the household's objective function to see this). Aggregating over generation t households to arrive at aggregate savings $S(t)$, we can use the capital accumulation equation to find

$$K(t+1) = S(t) = \frac{\beta}{1+\beta} L(t)w(t).$$

Using the firm's optimality condition, we can substitute for the wage:

$$K(t+1) = \frac{\beta}{1+\beta}(1-\alpha)A(t)K(t)^{\alpha}L(t)^{1-\alpha}$$

Note that with Cobb-Douglas production, the wage bill $L(t)w(t)$ is just a fraction $1-\alpha$ of total output $Y(t)$. The capital accumulation equation above is a *non-autonomous* first-order difference equation in capital: Given $K(t)$, it tells us how to calculate $K(t+1)$, but this calculation is time-varying because both $L(t)$ and $A(t)$ are growing. As in the Solow and neoclassical growth models, to characterize the steady-state we can attempt to write the capital accumulation equation as an *autonomous* first-order difference equation in a "detrended" or "normalized" capital-like variable. Start by dividing both sides by $L(t)$, using the assumption that $L(t+1) = (1+n)L(t)$:

$$\frac{K(t+1)}{L(t+1)} = \frac{1-\alpha}{1+n}\frac{\beta}{1+\beta}A(t)\left(\frac{K(t)}{L(t)}\right)^{\alpha}.$$

This is a first-order difference equation in the capital-labor ratio $K(t)/L(t)$, but it is again non-autonomous when there is technological progress ($g > 0$). We can perform the same "trick" again but with $A(t)$ by writing $A(t) = A(t)^{\frac{1}{1-\alpha}-\frac{\alpha}{1-\alpha}}$ and dividing each side by $A(t)^{\frac{1}{1-\alpha}}$:

$$\frac{K(t+1)}{A(t+1)^{\frac{1}{1-\alpha}}L(t+1)} = \frac{1}{(1+g)^{\frac{1}{1-\alpha}}}\frac{1-\alpha}{1+n}\frac{\beta}{1+\beta}\left(\frac{K(t)}{A(t)^{\frac{1}{1-\alpha}}L(t)}\right)^{\alpha}.$$

Here we also used the assumption that $A(t+1) = (1+g)A(t)$. We then arrive at an *autonomous* first-order difference equation in the capital-effective labor ratio $\tilde{k}(t) = K(t)/A(t)^{\frac{1}{1-\alpha}}L(t)$. Why does the technology shock $A(t)$ enter with the exponent $\frac{1}{1-\alpha}$? With Cobb-Douglas production, a Hicks-neutral shock $A(t)$ has precisely the same effect on the production technology as the labor-augment shock $A(t)^{\frac{1}{1-\alpha}}$. Uzawa's Theorem tells us that in any balanced growth path the capital stock must grow at the same rate as effective labor, which in this model is given by $A(t)^{\frac{1}{1-\alpha}}L(t)$. So it makes sense to choose this as a normalizing variable for the capital stock – and we know that it is the *right* choice because the capital accumulation equation becomes autonomous (i.e., stationary) when we do this.

To characterize the steady-state, we write the difference equation in terms of $\tilde{k}(t)$:

$$\tilde{k}(t+1) = \frac{1}{(1+g)^{\frac{1}{1-\alpha}}}\frac{1}{1+n}\frac{\beta}{1+\beta}(1-\alpha)\tilde{k}(t)^{\alpha}.$$

Any steady-state $\tilde{k}^*$ must satisfy this equation with $\tilde{k}(t+1) = \tilde{k}(t) = \tilde{k}^*$. The unique non-zero steady state is then

$$\tilde{k}^* = \left[\frac{1}{(1+g)^{\frac{1}{1-\alpha}}}\frac{1-\alpha}{1+n}\frac{\beta}{1+\beta}\right]^{\frac{1}{1-\alpha}}.$$

In this steady state, capital, output, and total consumption all grow at the same constant rate:

$$
\begin{aligned}
K(t) &= \tilde{k}^* A(t)^{\frac{1}{1-\alpha}} L(t) \\
&= (1+g)^{\frac{1}{1-\alpha}} (1+n) K(t-1), \\
Y(t) &= \left(\tilde{k}^*\right)^{\alpha} A(t)^{\frac{1}{1-\alpha}} L(t) \\
&= (1+g)^{\frac{1}{1-\alpha}} (1+n) Y(t-1), \\
C(t) &= Y(t) - K(t+1) \\
&= Y(t) - (1+g)^{\frac{1}{1-\alpha}} (1+n) K(t) \\
&= \left[1 - (1+g)^{\frac{1}{1-\alpha}} (1+n) \left(\tilde{k}^*\right)^{1-\alpha}\right] Y(t).
\end{aligned}
$$

In particular, all "per capita" variables $(Y(t)/L(t),\, K(t)/L(t),\, \ldots)$ grow at the constant rate $(1+g)^{\frac{1}{1-\alpha}}$. The wage similarly grows at the rate $(1+g)^{\frac{1}{1-\alpha}}$, and the interest rate is constant:

$$
\begin{aligned}
w(t) &= (1-\alpha) A(t) \left(\frac{K(t)}{L(t)}\right)^{\alpha} = (1-\alpha)\left(\tilde{k}^*\right)^{\alpha} A(t)^{\frac{1}{1-\alpha}}, \\
R(t) &= \alpha A(t) \left(\frac{L(t)}{K(t)}\right)^{1-\alpha} \qquad = \alpha \left(\tilde{k}^*\right)^{-(1-\alpha)}
\end{aligned}
$$

The steady state is globally stable provided that $\tilde{k}(t) \to \tilde{k}^*$ given any starting value $\tilde{k}(0)$. To show this, we can prove the stronger result that $\tilde{k}(t)$ converges *monotonically* to $\tilde{k}^*$: If $\tilde{k}(0) < \tilde{k}^*$, $\tilde{k}(t)$ increases at each t and converges to $\tilde{k}^*$, but if $\tilde{k}(0) > \tilde{k}^*$, $\tilde{k}(t)$ decreases at each t and converges to $\tilde{k}^*$. I prove this just when $\tilde{k}(0) < \tilde{k}^*$, because the argument is identical for the case with high initial capital. Write the difference equation for $\tilde{k}$ as

$$
\tilde{k}(t+1) = G\left(\tilde{k}(t)\right), \quad \text{where} \quad G\left(\tilde{k}\right) = \frac{1}{(1+g)^{\frac{1}{1-\alpha}}} \frac{1}{1+n} \frac{\beta}{1+\beta} (1-\alpha) \tilde{k}^{\alpha}.
$$

The function G is strictly increasing, and $\tilde{k}^*$ is the unique non-zero solution to the equation $\tilde{k} = G\left(\tilde{k}\right)$. With $\tilde{k}(t) < \tilde{k}^*$, these facts immediately imply

$$
\tilde{k}(t+1) = G\left(\tilde{k}(t)\right) < G\left(\tilde{k}^*\right) = \tilde{k}^*.
$$

Hence the capital-labor ratio is always bounded above by the steady-state value $\tilde{k}^*$. Moreover, the capital-labor ratio $\tilde{k}(t)$ is increasing over time:

$$
\tilde{k}(t+1) > \tilde{k}(t) \iff G\left(\tilde{k}(t)\right) > \tilde{k}(t) \iff \tilde{k}(t) < \tilde{k}^*.
$$

The final implication holds by direct calculation. Since $\tilde{k}(t)$ is strictly increasing and bounded above by the unique fixed point $\tilde{k}^*$ of G, we conclude that $\tilde{k}(t) \uparrow \tilde{k}^*$. Repeating the same argument for $\tilde{k}(0) > \tilde{k}^*$, we conclude that the steady-state equilibrium is globally stable.

Part 4. What is the effect of an increase in g on the equilibrium path?

Solution: An increase in g raises output per capita, the capital-labor ratio, consumption per capital, the wage, and the interest rate at each time t – not just in the steady state. To see this, recall from above the capital-labor ratio statisfies the non-autonomous first-order difference equation

$$\frac{K(t+1)}{L(t+1)} = \frac{1-\alpha}{1+n}\frac{\beta}{1+\beta}A(t)\left(\frac{K(t)}{L(t)}\right)^{\alpha}.$$

An increase in g raises $A(t)$ at each time $t > 0$. Since $K(0)/L(0)$ is fixed and the right side of this equation is increasing in $K(t)/L(t)$ and $A(t)$, we immediately observe that an increase in g raises $K(t)/L(t)$ at each time $t > 0$. This immediately implies the corresponding result for output per capita when we note

$$\frac{Y(t)}{L(t)} = A(t)\left(\frac{K(t)}{L(t)}\right)^{\alpha}.$$

The wage and interest rate satisfy

$$w(t) = (1-\alpha)A(t)\left(\frac{K(t)}{L(t)}\right)^{\alpha},$$

$$R(t) = \alpha\left(\frac{A(t)^{\frac{1}{1-\alpha}}L(t)}{K(t)}\right)^{1-\alpha}.$$

The same argument as above implies that $w(t)$ is increasing in g at each time $t > 0$. To determine the comparative static for the interest rate $R(t)$, recall that the capital-effective labor ratio satisfies the autonomous first-order difference equation

$$\tilde{k}(t+1) = \frac{1}{(1+g)^{\frac{1}{1-\alpha}}}\frac{1}{1+n}\frac{\beta}{1+\beta}(1-\alpha)\tilde{k}(t)^{\alpha}.$$

The right side of this equation is decreasing in g and increasing in $\tilde{k}(t)$, and the initial value $\tilde{k}(0)$ is fixed. These facts immediately imply that $\tilde{k}(t)$ is decreasing in g at each time $t > 0$. But since the interest rate is given by $R(t) = \alpha\tilde{k}(t)^{-(1-\alpha)}$, we observe that the interest rate at each time $t > 0$ is increasing in g. Finally, note that since both the wage $w(t)$ and the interest rate $R(t+1)$ are increasing in g, consumption for generation t while young $c_1(t)$ and while old $c_2(t+1)$ must both be increasing in g.

Part 5. In the rest of the question, assume that $g = 0$. Suppose that the equilibrium involves $r^* < n$. Explain why the equilibrium is referred to as "dynamically inefficient" in this case. Show that an unfunded Social Security system can increase the welfare of *all* future generations.

Solution: When $r^* = R^* - 1 < n$, the equilibrium is "dynamically inefficient" because the steady-state or limiting capital stock exceeds the golden rule capital stock that maximizes steady-state consumption. Intuitively, the equilibrium overaccumulates capital when $r^* < n$, and reducing the capital stock (equivalently, the quantity of savings) at each date would allow for an increase in consumption at each date. This implies that the equilibrium is *not Pareto optimal*, and we will provide a constructive proof to show that there are alternative allocations that strictly increase the welfare of all generations.

We proceed by introducing an unfunded Social Security system to the economy, which consists of a tax d on each generation while young and a corresponding transfer $(1+n)d$ to each generation while old. Since the population of each generation grows at rate n, this amounts to a mandatory transfer from young to old at each time t. I will show that for d sufficiently small, the equilibrium with the Social Security system Pareto dominates the equilibrium without the Social Security system.

The optimization problem for generation t now becomes

$$\max_{s(t)\in[0,w(t)-d]} \log\left(w(t)-d-s(t)\right)+\beta\log\left(R(t+1)s(t)+(1+n)d\right).$$

The interior first-order condition must be satisfied for d sufficiently small:

$$\frac{1}{w(t)-d-s(t)} = \frac{\beta R(t+1)}{R(t+1)s(t)+(1+n)d}.$$

Savings by generation t are then

$$s(t) = \frac{\beta}{1+\beta}w(t) - \frac{1}{1+\beta}\left(\beta + \frac{1+n}{R(t+1)}\right)d.$$

Note that with fixed prices $w(t)$ and $R(t+1)$, savings are always smaller when $d > 0$ as each household attempts to compensate for the lower consumption at t and the higher consumption at $t+1$ effected by the Social Security system. (But we have to see if this remains true in *general equilibrium* after prices adjust.) Aggregating across generation t households and using the representative firm's optimality conditions to substitute for the prices $w(t)$ and $R(t+1)$, we find that capital satisfies the non-autonomous first-order difference equation

$$K(t+1) = \frac{\beta}{1+\beta}(1-\alpha)K(t)^{\alpha}L(t)^{1-\alpha} - \frac{1}{1+\beta}\left[\beta + \frac{1+n}{\alpha}\left(\frac{K(t+1)}{L(t+1)}\right)^{1-\alpha}\right] dL(t)$$

Letting $k(t) = K(t)/L(t)$ denote the capital-labor ratio, we can divide through by $L(t)$ and rearrange to find the autonomous first-order difference equation

$$(1+n)(1+\beta)k(t+1) + \left[\beta + \frac{1+n}{\alpha}k(t+1)^{1-\alpha}\right]d = \beta(1-\alpha)k(t)^{\alpha}$$

The second term on the left side is the new term that arises with $d > 0$. Just as in the standard OLG model, this difference equation fully characterizes the equilibrium with the Social Security system: All quantities and prices at each time can be written as a function of the capital-labor ratio $k(t)$ and the (exogenous) number of workers $L(t)$. However, when $d > 0$ we cannot generally solve for $k(t+1)$ as a function of $k(t)$ in closed form. But since the left side of this equation is strictly increasing in $k(t+1)$, tends to zero as $k(t+1) \to 0$, and tends to infinity as $k(t+1) \to \infty$, there is still a unique solution $k(t+1)$. To perform comparative statics with respect to d, we write $k(t,d)$ to emphasize the dependence of the sequence of capital-labor ratios on d, and we write the difference equation in more compact form as

$$H(k(t+1,d),d) = \beta(1-\alpha)k(t,d)^{\alpha}.$$

We can implicitly differentiate with respect to d to find

$$k_d(t+1,d) = \frac{\beta(1-\alpha)\alpha k(t,d)^{\alpha-1}k_d(t,d) - H_d(k(t+1,d),d)}{H_k(k(t+1,d),d)}.$$

I claim that this equation implies $\partial k(t+1,d)/\partial d < 0$. Since $H_d > 0$ and $H_k > 0$, this holds provided that $k_d(t,d) < 0$. But at $t = 0$, since $k(0,d) = k(0)$ is exogenously fixed, this equation reduces to

$$k_d(1,d) = -\frac{H_d(k(1,d),d)}{H_k(k(1,d),d)} < 0.$$

By induction, we can conclude that $k_d(t+1,d) < 0$ for $t \geq 0$.

Thus far, we have established that expanding the Social Security system (i.e., raising d) lowers the capital-labor ratio at each time in the (unique) competitive equilibrium. How does this fact help us show that we can achieve a Pareto improvement by introducing the Social Security system? The idea is that a small increase from $d = 0$ to $d > 0$ reduces the capital-labor ratio at each time, and since the economy overaccumulates capital in the $d = 0$ equilibrium, this adjustment could (and in fact will!) raise consumption at each time by reducing the overaccumulation. For our purposes, it will suffice to show that each generation's budget constraint slackens when we raise d marginally from $d = 0$ to $d > 0$. For arbitrary $d > 0$, generation t's budget constraint is

$$c_1(t) \leq w(t,d) + \left(\frac{1+n}{R(t+1,d)} - 1\right) d - \frac{c_2(t+1)}{R(t+1,d)}.$$

It suffices to show that, when $c_1(t)$ and $c_2(t+1)$ take their steady-state equilibrium values with $d = 0$, the right side of this inequality is increasing in d near $d = 0$. Recall that these steady-state equilibrium values are given by

$$c_1^* = w^* - s^* = w^* - (1+n)\,k^*,$$
$$c_2^* = R^*\,(1+n)\,k^*.$$

The right-hand side (RHS) of the inequality above can then be written

$$\text{RHS}(t,d) = w(t,d) + \left(\frac{1+n}{R(t+1,d)} - 1\right) d - \frac{R^*\,(1+n)\,k^*}{R(t+1,d)}$$

Differentiating with respect to d and evaluating at $d = 0$, we have

$$\text{RHS}_d(t,0) = w_d(t,0) + \frac{1+n}{R^*} - 1 + \frac{(1+n)\,k^*}{R^*} R_d(t+1,0),$$

where we used the identity $R(t+1,0) = R^*$. To calculate the derivatives $w_d(t,0)$ and $R_d(t+1,0)$, we make use of the equilibrium price conditions

$$w(t,d) = (1-\alpha)\,k(t,d)^{\alpha},$$
$$R(t+1,d) = \alpha k(t+1,d)^{-(1-\alpha)}.$$

Differentiating yields

$$w_d(t,d) = \alpha\,(1-\alpha)\,k(t,d)^{-(1-\alpha)} k_d(t,d),$$
$$R_d(t+1,d) = -\alpha\,(1-\alpha)\,k(t+1,d)^{-(2-\alpha)} k_d(t+1,d).$$

We can then write

$$\text{RHS}_d(t,0) = \frac{1+n}{R^*} - 1 + \alpha(1-\alpha)(k^*)^{-(1-\alpha)}\left[k_d(t,0) - \frac{1+n}{R^*}k_d(t+1,0)\right].$$

Our final observation is that the evolution equation for $k_d(t+1,0)$ implies that $k_d(t+1,0) < k_d(t,0)$ for $t \geq 0$. Intuitively, starting from the $d = 0$ steady-state capital-labor ratio k^*, introducing the unfunded Social Security system $d > 0$ requires a reduction in the capital-labor ratio at each time to reach the new and lower steady-state capital-labor ratio $k^*(d)$. The capital-labor ratio declines monotonically to $k^*(d)$, so in response to the introduction of $d > 0$, the capital-labor ratio at $t+1$ must fall more than at t relative to the initial steady-state k^*. This observation implies

$$\begin{aligned}\text{RHS}_d(t,0) &> \frac{1+n}{R^*} - 1 + \alpha(1-\alpha)(k^*)^{-(1-\alpha)}\left[k_d(t+1,0) - \frac{1+n}{R^*}k_d(t+1,0)\right] \\ &= \left(1 - \alpha(1-\alpha)(k^*)^{-(1-\alpha)}k_d(t+1,0)\right)\left(\frac{1+n}{R^*} - 1\right)\end{aligned}$$

The first factor is positive since $k_d(t+1,0) < 0$, and the second factor is positive since the initial steady-state equilibrium is dynamically inefficient $(1+n > R^*)$. Hence $\text{RHS}_d(t,0) > 0$, and since preferences are locally non-satiated, introducing a (small) unfunded Social Security system must strictly raise welfare for each generation in equilibrium.[1]

Part 6. Show that if $r^* > n$, then any unfunded Social Security system that increases the welfare of the current old generation must reduce the welfare of some future generation.

Solution: In this case, we can directly apply the First Welfare Theorem with a countably infinite number of households (see problem set solutions for details).

[1] Technically I've shown this only for generations $t \geq 0$, but it is immediate that generation $t = -1$ (i.e., the initial elderly who own the initial capital stock) strictly benefits from the new transfer of $(1+n)d$ per capita.

2 Romer (1990)

2.1 Setup

This model exists in continuous time $t \in [0, \infty)$ and consists of a representative household with labor endowment $L(t) = L\exp(nt)$, discount rate $\rho > 0$, and consumption utility $u(c) = c^{1-\theta}/(1-\theta)$. A unique final good (and numeraire) is produced at each time t using the Cobb-Douglas production technology

$$Y(t) = \frac{1}{1-\beta}\left[\int_0^{N(t)} x(\nu,t)^{1-\beta} d\nu\right] L_E(t)^{\beta},$$

where $L_E(t)$ denotes the quantity of labor employed in final good production, $x(\nu,t)$ denotes the quantity of intermediate good ν used in final good production, and $N(t)$ denotes the number of intermediate varieties discovered up to time t. Each intermediate is produced using the final good at marginal cost $\psi > 0$, and intermediates are assumed to depreciate completely at each time.

Labor can also be used to conduct research and development (R&D) for the discovery of new intermediate varieties. Given a quantity of labor input $L_R(t)$, the number of varieties increases according to the evolution equation

$$\dot{N}(t) = N(t)^{\phi}\eta L_R(t).$$

Here $\phi \leq 1$ controls the strength of knowledge spillovers across time: With $\phi > 0$, greater existing knowledge makes current researchers more productive in the discovery of new varieties, and this effect is stronger when ϕ is larger. I restrict $\phi \leq 1$ so that we do not obtain "explosive" growth even when $L_R(t)$ is constant over time. For reasons that will become clear below, I refer to the case with $\phi = 1$ as exhibiting *dynamic constant returns to R&D*, while the case with $\phi < 1$ exhibits *dynamic decreasing returns to R&D*.

Labor is allocated between final good production and R&D according to profit-maximizing behavior by two different kinds of firms. A representative final good producer chooses the quantities of all inputs ($x(\nu,t)$ for $\nu \in [0, N(t)]$ and $L_E(t)$) to maximize final output, taking the price of each intermediate $p(\nu,t)$ and the wage $w(t)$ as given. A large mass of firms also employ labor to discover new intermediate varieties. Each of these "potential monopolists" can employ one unit of labor to discover a new variety at rate $N(t)^{\phi}\eta$.[2] Aggregating across all potential

[2] I write this as if each potential monopolist can only employ one unit of labor for R&D, but since the "production technology for knowledge" $\dot{N} = N^{\phi}\eta L_R$ exhibits constant returns to scale in L_R, it's all the same if each potential

monopolists that employ labor, the total flow rate of new ideas is then $\dot{N}(t) = N(t)^{\phi}\eta L_R(t)$. Potential monopolists find it optimal to employ labor for R&D provided that the value $V(t)$ of discovering a new variety at t dominates the cost of discovery. Equivalently, this holds when the value of employing an additional unit of labor at wage $w(t)$ is weakly smaller than the value generated by that labor, which equals the flow rate of discovery $N(t)^{\phi}\eta$ times the value $V(t)$. In equilibrium, potential monopolists continue to enter until the wage $w(t)$ is driven up to this flow value $N(t)^{\phi}\eta V(t)$, so that we satisfy

$$N(t)^{\phi}\eta V(t) \leq w(t) \quad \text{and} \quad L_R(t) \geq 0,$$

with complementary slackness.

To complete the description of the model, we must determine the value $V(t)$. I assume that each monopolist that successfully invents a new intermediate variety ν receives a perpetual patent on that variety. As a result, it can set its price $p(\nu, t)$ at each time t to maximize profits, taking all remaining equilibrium objects except for the quantity $x(\nu, t)$ as given. Letting $\pi(t)$ denote the profits at each time t, and noting that π does not depend on ν because all existing intermediates $\nu \in [0, N(t)]$ enter final production symmetrically and have the same marginal cost ψ, the value $V(t)$ must satisfy

$$V(t) = \int_t^{\infty} \exp\left(-\int_t^s r(u)du\right)\pi(s)ds.$$

Here $r(t)$ denotes the equilibrium interest rate at time t. The value of ownership of an intermediate is then the present discounted value of all future profit flows, discounted to present using the "market" discount rate $r(t)$. Differentiating with respect to t implies that this value also satisfies the Hamilton-Jacobi-Bellman (HJB) equation

$$r(t)V(t) = \pi(t) + \dot{V}(t).$$

This equation expresses the "arbitrage condition" that the instantaneous return to owning an intermediate $r(t)V(t)$ must equal the flow dividend $\pi(t)$ plus any "capital gains" $\dot{V}(t)$.

Finally, note that in this version of the Romer (1990) model, the household can "save" only in a fairly implicit way. Just as in the neoclassical growth model, we allow the household access to an asset $\mathcal{A}(t)$ that pays an instantaneous return $r(t)$ at each time t and, from the household's perspective, allows it to transfer consumption across time. The household's opti-

monopolist can employ any quantity of labor it wishes.

mal consumption stream can again be summarized by the Euler equation and the transversality condition

$$\frac{\dot{c}(t)}{c(t)} = \frac{1}{\theta}(r(t)-\rho),$$
$$0 = \lim_{t\to\infty} \exp\left(-\int_0^t r(s)ds\right)\mathcal{A}(t).$$

But how does "saving" actually happen, and what is the asset $\mathcal{A}(t)$ since this model does not have physical capital? In equilibrium, the household's assets at each time t must be equal to the value of all intermediate monopolists: $\mathcal{A}(t) = N(t)V(t)$. Intuitively, when the household wants to transfer consumption into the future, the economy responds by reducing the quantity of labor $L_E(t)$ employed in final good production and raising the quantity of labor $L_R(t)$ employed in R&D. This raises the rate at which new intermediates are discovered and hence the "supply" of assets $N(t)V(t)$. As we will see below, this works to raise consumption in the future by making labor more productive in producing the final good, which increases consumption (holding the labor input fixed).

The way this works in equilibrium is as follows: Fix a path for per capita consumption $[c(t)]_{t\geq 0}$, and note that the interest rate $r(t)$ is pinned down at each time by the household's Euler equation. Suppose we temporarily increase the household's desire for saving at time t (say, by reducing ρ temporarily). This leads the household to demand more assets $\mathcal{A}(t)$, which places downward pressure on the interest rate $r(t)$. But this raises the discounted present value $V(t)$ of future profits earned by an intermediate monopolist, stimulating additional entry and increasing the rate of production of new assets $\dot{N}(t)$.[3] This can only happen if labor is reallocated away from final good production and toward R&D, which reduces present consumption in favor of future consumption.

2.2 Static Equilibrium Conditions

Before studying the dynamic equilibrium in this model, we can make some progress by studying the static equilibrium conditions of the final good producer and the monopolists of existing intermediates $\nu \in [0, N(t)]$. Given the wage $w(t)$ and the intermediate prices $[p(\nu,t)]_{\nu=0}^{N(t)}$, the final good producer chooses $L_E(t)$ and $[x(\nu,t)]_{\nu=0}^{N(t)}$ to maximize profits. The first-order

[3]This explanation assumes that initially $\dot{N}(t) > 0$. If we are instead in an equilibrium with no R&D, then the interest rate is the only part of the equilibrium that adjusts to the increased propensity to save, ensuring that the household finds it optimal to consume according to the original consumption path at each time. This adjustment is just as in the Lucas asset pricing model, and I'm happy to explain further if helpful.

optimality conditions are

$$w(t) = \beta \frac{Y(t)}{L_E(t)},$$
$$p(\nu, t) = L_E(t)^{\beta} x(\nu, t)^{-\beta}.$$

We will eventually use the first condition to determine the wage $w(t)$. The second condition defines the (inverse) demand curve observed by each intermediate monopolist ν. Given this demand curve, the monopolist chooses the price $p(\nu, t)$ to maxmize its own profits at t:

$$\max_p (p - \psi) L_E(t) p^{-1/\beta}.$$

The solution to this problem is

$$p(\nu, t) = \frac{1}{1-\beta}\psi,$$

with corresponding quantity and profits

$$x(\nu, t) = \bar{x} L_E(t), \quad \text{where} \quad \bar{x} = \left(\frac{\psi}{1-\beta}\right)^{-\frac{1}{\beta}}$$
$$\pi(t) = \bar{\pi} L_E(t), \quad \text{where} \quad \bar{\pi} = \beta\left(\frac{\psi}{1-\beta}\right)^{-\frac{1-\beta}{\beta}}.$$

Total output then satisfies

$$Y(t) = \frac{\bar{x}^{1-\beta}}{1-\beta} N(t) L_E(t),$$

so that the wage becomes

$$w(t) = \beta \frac{Y(t)}{L_E(t)} = \beta \frac{\bar{x}^{1-\beta}}{1-\beta} N(t).$$

Finally, total consumption is

$$\begin{aligned} C(t) &= Y(t) - \psi \int_0^{N(t)} x(\nu, t) d\nu \\ &= Y(t) - \psi \bar{x} N(t) L_E(t) \\ &= \left(1 - (1-\beta)^{1+\beta}\right) Y(t). \end{aligned}$$

Note that the crucial feature of this model is that final output $Y(t)$ is proportional to $N(t)$ and $L_E(t)$: $N(t)$ acts like labor-augmenting technological progress, so provided that $L_E(t)$ eventually settles to a constant value, we expect to achieve constant growth in output per capita if $N(t)$ increases at a constant rate.

2.3 Dynamic Constant Returns: $\phi = 1$, $n = 0$

To characterize the equilibrium with $\phi = 1$ and $n = 0$, I begin as usual by studying the balanced growth path. Suppose an equilibrium in which output and consumption grow at the constant rate $g \geq 0$. The household's Euler equation then implies that the interest rate is constant and satisfies the standard "Ramsey formula"

$$r^* = \rho + \theta g.$$

There are two cases to consider: Either $\dot{N}(t) = 0$ always, or $\dot{N}(t) > 0$ at some time t. I consider these cases in turn.

Case 1: $\dot{N}(t) \equiv 0$. In this case, we must have $L_E(t) = L$ at each time t, so that the economy permanently stagnates with output $Y(t) = \frac{\bar{x}^{1-\beta}}{1-\beta} N(0)L$, wage $w(t) = \beta \frac{\bar{x}^{1-\beta}}{1-\beta} N(0)$, and interest rate $r^* = \rho$. To ensure that this is a valid equilibrium, we must only check that potential monopolists find it weakly optimal not to conduct R&D. The value $V(t)$ of an intermediate is $V(t) = \bar{\pi} L/\rho$, so that free entry with $L_R(t) = 0$ requires

$$N(0)\eta V(t) \leq w(0) \iff \rho \geq \eta(1-\beta)L.$$

Under this parameter restriction, the economy has a balanced growth path with $\dot{N}(t) \equiv 0$.

Case 2: $\dot{N}(t) > 0$ at some t. In this case, we must have $L_R(t) > 0$, so that the free-entry condition implies

$$N(t)\eta V(t) = w(t) \Rightarrow \eta V(t) = \beta \frac{\bar{x}^{1-\beta}}{1-\beta}.$$

The implication follows from the characterization of the wage $w(t)$ above. Hence $V(t) = V^*$ is constant,[4] and the HJB equation for $V(t)$ implies

$$V^* = \frac{\bar{\pi} L_E(t)}{r^*}.$$

[4]Here I'm really assuming that $\dot{N}(t) > 0$ on an interval or that $\dot{N}(t)$ is continuous, both of which are innocuous along a balanced growth path.

But then $L_E(t)$ must be constant, $L_E(t) \equiv L_E^*$. This constant quantity of labor employed in final good production and the growth rate g must satisfy the system of equations

$$
\begin{aligned}
L_E^* &= r^* \frac{V^*}{\bar{\pi}} = \frac{\rho + \theta g}{\eta (1-\beta)}, \\
g &= \frac{\dot{N}(t)}{N(t)} = \eta \left(L - L_E^* \right).
\end{aligned}
$$

The solution is

$$
\begin{aligned}
L_E^* &= \frac{1}{\eta} \frac{\theta \eta L + \rho}{1 - \beta + \theta}, \\
g &= \frac{\eta (1-\beta) L - \rho}{1 - \beta + \theta}.
\end{aligned}
$$

To ensure that we have characterized a valid equilibrium, we must check that (i) the free-entry condition is satisfied with $g > 0$ and (ii) $r^* > g$, so that the equilibrium features finite expected discounted output. The first condition holds provided that $\rho < \eta(1-\beta)L$, and the second condition holds provided that

$$
\rho > (1-\theta) g \iff \rho > \frac{\eta (1-\theta)(1-\beta) L}{2-\beta}
$$

The analysis above provides a full characterization of the unique balanced growth path in this economy. What about transitional dynamics? We shouldn't expect any in this model, because the economy does not have any "concave" features (like diminishing marginal returns to an accumulating factor) that would yield sluggish adjustment to the balanced growth path. To see this, suppose the balanced growth path features positive growth,and note that the free-entry condition and Euler equations imply the following characterizations of the interest rate $r(t)$:

$$
\begin{aligned}
r(t) &= \rho + \theta \left[\frac{\dot{N}(t)}{N(t)} + \frac{\dot{L}_E(t)}{L_E(t)} \right] \\
&= \rho + \theta \left[\eta (L - L_E(t)) + \frac{\dot{L}_E(t)}{L_E(t)} \right], \\
r(t) &= \eta(1-\beta) L_E(t).
\end{aligned}
$$

These equations imply

$$L_E(t) - L_E^* = \frac{1}{\eta} \frac{\theta}{1-\beta+\theta} \frac{\dot{L}_E(t)}{L_E(t)}$$

When $L_E(t)$ is above its BGP value L_E^*, this equation implies that the growth rate of $L_E(t)$ is positive. If ever $L_E(t) > L_E^*$, the unique solution to this differential equation would feature $L_E(t) \to \infty$, violating the labor market clearing condition $L_E(t) + L_R(t) \leq L$. Similarly, if ever $L_E(t) < L_E^*$, the solution would feature $L_E(t) \to 0$, implying no consumption as $t \to \infty$ and violating the household's transversality condition. We conclude that any equilibrium must feature $L_E(t) = L_E^*$ at all times t, so that we immediately follow the balanced growth path.

2.4 Dynamic Decreasing Returns: $\phi < 1$, $n > 0$

Weakening knowledge spillovers by reducing ϕ below 1 has (surprisingly!) strong implications for growth in this model. For example, if the number of workers allocated to R&D is held fixed, it is easy to see that the growth rate of labor productivity will tend to zero over time:

$$\frac{\dot{N}(t)}{N(t)} = \eta N(t)^{\phi-1} L_R \to 0,$$

where the limit holds since $\phi < 1$.[5] Conversely, when $\phi = 1$, the growth rate of labor productivity *diverges* if the number of workers allocated to R&D increases at a constant rate (e.g., because of population growth):

$$\frac{\dot{N}(t)}{N(t)} = \eta L_R(t) \to \infty.$$

This is the essence of the *scale effect* in the model with $\phi = 1$: The growth rate on the balanced growth path is increasing in the *quantity* of labor L. As Jones (1995) discusses, this scale effect is counterfactual across many countries and time periods, and that paper proposes a variation of the Romer (1990) model with $\phi < 1$ but $n > 0$ to remove it.

In a balanced growth path with a constant *share* of workers s allocated to R&D, it is straightforward to determine the growth rate g of labor productivity (or output per worker). Since

[5] This is why I think of this case as encapsulating "dynamic decreasing returns": The marginal improvement in R&D productivity from additional knowledge accumulation decays over time.

output per worker is proportional to $N(t)$, g must satisfy

$$g = \frac{\dot{N}(t)}{N(t)} = \eta N(t)^{\phi-1} s L(t).$$

Log differentiating both sides implies

$$0 = \frac{\dot{L}(t)}{L(t)} - (1-\phi)\frac{\dot{N}(t)}{N(t)} = n - (1-\phi)\, g \iff g = \frac{n}{1-\phi}.$$

Hence the long-run growth rate of output per worker is entirely determined by the growth rate of the population and the extent of dynamic decreasing returns to knowledge accumulation ϕ. Using similar arguments to those as in the $\phi = 1$ case, it is straightforward to characterize the remaining equilibrium objects along the balanced growth path when $\phi < 1$.