14.452 Recitation 5: Directed Technical Change

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Recitation Plan: Solve the canonical directed technical change model (Acemoglu 1998, 2002)

1 Setup

The baseline directed technical change model generalizes the Romer (1990) "lab equipment" model by including two primary factors (L and H) that are used with factor-specific machines to produce two intermediate goods (Y_L and Y_H). These intermediates are then aggregated to produce final output.

This model exists in continuous time $t \in [0, \infty)$ and consists of a representative household with constant endowment of "low-skill labor" L, constant endowment of "high-skill labor" H, discount rate $\rho > 0$, and consumption utility $u(C) = C^{1-\theta}/(1-\theta)$. A unique final good (and numeraire) is produced at each time t using the CES production technology

$$Y(t) = \left[\gamma_L Y_L(t)^{\frac{\varepsilon-1}{\varepsilon}} + \gamma_H Y_H(t)^{\frac{\varepsilon-1}{\varepsilon}}\right]^{\frac{\varepsilon}{\varepsilon-1}},$$

where $Y_L(t)$ and $Y_H(t)$ are intermediate goods produced according to the Cobb-Douglas production technologies

$$Y_i(t) = \frac{1}{1-\beta}\left(\int_0^{N_i(t)} x_i(\nu, t)^{1-\beta}\, d\nu\right) Q_i^{\beta},$$

where $i \in \{L, H\}$, $Q_L = L$, and $Q_H = H$. Here $x_i(\nu, t)$ denotes the quantity of "machine" (i, ν) used in final production, and $N_i(t)$ denotes the number of type i machine varieties discovered up to time t. Each type i machine is produced using the final good at marginal cost $\psi > 0$, and machines are assumed to depreciate completely after use. Note in particular that each intermediate Y_L and Y_H uses a different set of machines in production.

The final good can also be used to fund R&D for the discovery of new machine varieties. Given investment $Z_i(t)$ in R&D for intermediate i, the number of new varieties for i increases according to the evolution equation

$$\dot{N}_i(t) = \eta_i Z_i(t).$$

This specification of the "innovation possibilities frontier" (or the production technology for innovations) implies that R&D is *fully directed*: Any investment in R&D for i can only lead to new innovations for intermediate i. More generally, we might imagine that sometimes R&D for i happens to lead to an innovation for a new machine for the other intermediate $-i$. There are some "second generation" models of directed technical change in which this possibility plays an important role.

Final production is allocated between consumption and R&D according to profit-maximizing behavior by three different kinds of firms. A representative final good producer chooses the quantities of the intermediates $Y_L(t)$ and $Y_H(t)$ to maximize profits, taking the corresponding prices $p_L(t)$ and $p_H(t)$ as given. Each intermediate i has a representative producer that chooses the quantity of each machine $x_i(\nu, t)$ for $\nu \in [0, N_i(t)]$ and the quantity of its primary factor $Q_i(t)$ to maximize its own profits, taking the price of the intermediate $p_i(t)$, the prices of machines $p_i(\nu, t)$, and the price of the factor $w_i(t)$ as given. Finally, a large mass of firms invest the final good into R&D to produce new machine varieties. Each of these "potential monopolists" can employ one unit of the final good to discover a new variety for intermediate i at rate η_i.[1] Aggregating across all potential monopolists for each intermediate, the total flow rate of new machines for i is then $\dot{N}_i(t) = \eta_i Z_i(t)$. Potential monopolists find it optimal to invest in R&D for i provided that the value $V_i(t)$ of discovering a new variety at t dominates the cost of discovery. Equivalently, this holds when the value of investing an additional unit of final output is weakly smaller than the value generated by that investment, which equals the flow rate of discovery η_i times the value $V_i(t)$. In equilibrium, potential monopolists continue to enter until the marginal benefit of investment $\eta_i V_i(t)$ is driven down to the marginal cost 1, so that

$$\eta_i V_i(t) \leq 1 \quad \text{and} \quad Z_i(t) \geq 0,$$

with complementary slackness.

To complete the description of the model, we must determine the value $V_i(t)$. I assume that each monopolist that successfully invents a new machine variety (i, ν) receives a perpetual

[1] I write this as if each potential monopolist can only employ one unit of labor for R&D, but since the "production technology for knowledge" $\dot{N}_i = \eta_i Z_i$ exhibits constant returns to scale in Z_i, it's all the same if each potential monopolist can employ any quantity of labor it wishes.

patent on that variety. As a result, it can set its price $p_i(\nu, t)$ at each time t to maximize profits, taking all remaining equilibrium objects except for the quantity $x_i(\nu, t)$ as given. Letting $\pi_i(t)$ denote the profits at each time t, and noting that π_i does not depend on ν because all existing machines $\nu \in [0, N_i(t)]$ enter intermediate production symmetrically and have the same marginal cost ψ, the value $V_i(t)$ must satisfy

$$V_i(t) = \int_t^\infty \exp\left(-\int_t^s r(u)du\right)\pi_i(s)ds.$$

Here $r(t)$ denotes the equilibrium interest rate at time t. The value of ownership of an machine is then the present discounted value of all future profit flows, discounted to present using the "market" discount rate $r(t)$. Differentiating with respect to t implies that this value also satisfies the Hamilton-Jacobi-Bellman (HJB) equation

$$r(t)V_i(t) = \pi_i(t) + \dot{V}_i(t).$$

This equation expresses the "arbitrage condition" that the instantaneous return to owning an intermediate $r(t)V_i(t)$ must equal the flow dividend $\pi_i(t)$ plus any "capital gains" $\dot{V}_i(t)$.

Note in particular that the interest rate $r(t)$ *does not depend on* i. This holds by an arbitrage argument: Suppose we allowed the household access to two assets, $\mathcal{A}_L(t)$ and $\mathcal{A}_H(t)$, that it can freely buy or sell at each time t. In equilibrium, the quantity of asset i held by the household must equal the corresponding supply, which is just the total value of all machine monopolists for intermediate i: $\mathcal{A}_i(t) = N_i(t)V_i(t)$. If the two assets offered different rates of return $r_L(t) \neq r_H(t)$ at some time t, the household could take advantage of an arbitrage trade, selling the asset with the smaller rate of return and buying the asset with the larger rate of return so as to slacken the future budget constraint. Since the household takes the rates of return $r_L(t)$ and $r_H(t)$ as fixed, it would seek to sell an infinite quantity of the more "expensive" asset and buy an infinite quantity of the "cheaper" asset, which violates market clearing. Equilibrium in the asset markets then requires $r_L(t) = r_H(t) \equiv r(t)$, so we can treat the two assets $\mathcal{A}_L(t)$ and $\mathcal{A}_H(t)$ as essentially one asset $\mathcal{A}(t)$ with rate of return $r(t)$. The household's optimal consumption stream can again be summarized by the Euler equation and the transversality condition

$$\begin{aligned}\frac{\dot{C}(t)}{C(t)} &= \frac{1}{\theta}(r(t) - \rho),\\ 0 &= \lim_{t\to\infty} \exp\left(-\int_0^t r(s)ds\right)\mathcal{A}(t).\end{aligned}$$

In equilibrium, the household's assets at each time t must be equal to the value of all machine monopolists: $\mathcal{A}(t) = N_L(t)V_L(t) + N_H(t)V_H(t)$. Intuitively, when the household wants to transfer consumption into the future, the economy responds by reducing the quantity of final output for consumption $C(t)$ and raising the quantity of final output invested in R&D. This raises the rate at which new intermediates are discovered and hence the "supply" of assets $N_L(t)V_L(t) + N_H(t)V_H(t)$. As we will see below, this works to raise consumption in the future by making the factors L and H more productive, which increases consumption (holding future R&D investment fixed).

2 Analysis

2.1 Static Equilibrium Conditions

Before studying the dynamic equilibrium in this model, we can make some progress by studying the static equilibrium conditions of the final good producer, the intermediate producers, and the monopolists of existing intermediates (i, ν) for $\nu \in [0, N_i(t)]$.

Final good producer: Given the prices $p_L(t)$ and $p_H(t)$ for intermediates, the final good producer chooses $Y_L(t)$ and $Y_H(t)$ to maximize profits. The corresponding first-order conditions are

$$p_i(t) = \gamma_i \left(\frac{Y_i(t)}{Y(t)}\right)^{-\frac{1}{\varepsilon}}, \quad i \in \{L, H\}.$$

Let $p(t) = p_H(t)/p_L(t)$ denote the intermediate price ratio, and let $\gamma = \gamma_H/\gamma_L$. We can divide the two conditions above to find

$$p(t) = \gamma \left(\frac{Y_H(t)}{Y_L(t)}\right)^{-\frac{1}{\varepsilon}}.$$

This condition is quite important for the analysis that follows, because it relates the relative price of the two intermediates to the relative quantities employed in final production. This relationship is mediated by the elasticity of substitution ε: The cost-minimizing input ratio $Y_H(t)/Y_L(t)$ responds more strongly to the price ratio $p(t)$ when ε is large. In some sense, this relative price condition is the only "interesting" first-order condition from the final good producer's problem. The remaining first-order condition simply implies that the final good producer makes zero profits, so that its "ideal price" (unit cost) is equal to the price of final output (normalized to 1):

$$1 = \left[\gamma_L^{\varepsilon} p_L(t)^{1-\varepsilon} + \gamma_H^{\varepsilon} p_H(t)^{1-\varepsilon}\right]^{\frac{1}{1-\varepsilon}}.$$

Dividing through by $p_L(t)$, this condition allows us to recover the level of intermediate prices $p_L(t)$ and $p_H(t)$ from the price ratio $p(t)$:

$$p_L(t) = \left[\gamma_L^{\varepsilon} + \gamma_H^{\varepsilon} p(t)^{1-\varepsilon}\right]^{-\frac{1}{1-\varepsilon}}.$$

Intermediate producer: Given the intermediate price $p_i(t)$, the prices $p_i(\nu, t)$ of each type i machine $\nu \in [0, N_i(t)]$, and the wage $w_i(t)$ for factor i, the type i intermediate producer chooses the quantities $Q_i(t)$ and $[x_i(\nu, t)]_{\nu=0}^{N_i(t)}$ to maximize profits. The first-order optimality conditions are

$$w_i(t) = \beta \frac{p_i(t) Y_i(t)}{Q_i},$$
$$p_i(\nu, t) = p_i(t) Q_i^{\beta} x_i(\nu, t)^{-\beta}.$$

Here I make use the factor market-clearing condition to write $Q_i(t) = Q_i$, where $Q_L = L$ and $Q_H = H$. We will eventually use the first condition to determine the wage $w_i(t)$ for factor i. The second condition defines the (inverse) demand curve observed by each machine producer (i, ν). The key difference from the one-sector Romer (1990) model is that the intermediate price $p_i(t)$ appears in the first-order conditions, and in particular in the inverse demand curve for each machine producer – more on this below.

Machine producer: Given the inverse demand curve $p_i(\nu, t) = p_i(t) Q_i^{\beta} x_i(\nu, t)^{-\beta}$, the monopolist (i, ν) chooses the price $p_i(\nu, t)$ to maxmize its own profits at t:

$$\max_{p_i(\nu,t)} (p_i(\nu, t) - \psi) Q_i \left(\frac{p_i(\nu, t)}{p_i(t)}\right)^{-1/\beta}.$$

The solution to this problem is

$$p_i(\nu, t) = \frac{1}{1-\beta}\psi,$$

with corresponding quantity and profits

$$x_i(\nu,t) = \bar{x} p_i(t)^{1/\beta} Q_i, \quad \text{where} \quad \bar{x} = \left(\frac{\psi}{1-\beta}\right)^{-\frac{1}{\beta}}$$
$$\pi_i(t) = \bar{\pi} p_i(t)^{1/\beta} Q_i, \quad \text{where} \quad \bar{\pi} = \beta\left(\frac{\psi}{1-\beta}\right)^{-\frac{1-\beta}{\beta}}.$$

Note two interesting features of the profit function $\pi_i(t)$: First, profits are increasing in the quantity of the corresponding factor Q_i. This *market size effect* incentivizes R&D directed toward (machines that can be used by) the more abundant factor. But profits are also increasing in $p_i(t)$ because each type i machine is used more intensively when the price of intermediate i is larger. This *price effect* incentivizes R&D directed toward the factor with the larger corresponding intermediate price. We will see that these effects are offsetting in equilibrium, because with fixed $N_L(t)$ and $N_H(t)$ there is a natural inverse relationship between $p_i(t)$ and Q_i. Which effect dominates crucially depends on the elasticity of substitution ε and the labor share β.

Putting things together: Given our expression for the quantity of machine (i, ν) used in equilibrium, we find that total output of intermediate i must satisfy

$$Y_i(t) = \frac{\bar{x}^{1-\beta}}{1-\beta} p_i(t)^{\frac{1-\beta}{\beta}} N_i(t) Q_i.$$

The wage for factor i is then

$$w_i(t) = \beta \frac{\bar{x}^{1-\beta}}{1-\beta} p_i(t)^{\frac{1}{\beta}} N_i(t).$$

The total quantity of final output used in the production of machines is

$$X(t) = \bar{x}\left[N_L(t) p_L(t)^{\frac{1}{\beta}} L + N_H(t) p_H(t)^{\frac{1}{\beta}} H\right].$$

Using our equation for the intermediate price ratio from the final good producer's solution, we also have

$$p(t) = \gamma\left(p(t)^{\frac{1-\beta}{\beta}} \frac{N_H(t) H}{N_L(t) L}\right)^{-\frac{1}{\varepsilon}}.$$

The assumption of a CES production technology for the final good allows us to solve this equation explicitly for $p(t)$ as a function of exogenous objects and the state variables ($N_L(t)$ and

$N_H(t)$):

$$p(t) = \gamma^{\frac{\beta\varepsilon}{\sigma}} \left(\frac{N_H(t)H}{N_L(t)L} \right)^{-\frac{\beta}{\sigma}},$$

where $\sigma = 1 + \beta(\varepsilon - 1)$. Using the equilibrium condition for the wage $w_i(t)$ derived above, we can use this expression for $p(t)$ to find an expression for the relative wage $\omega(t) = w_H(t)/w_L(t)$ as a function of exogenous objects and the state variables:

$$\begin{aligned} \omega(t) &= p(t)^{\frac{1}{\beta}} \frac{N_H(t)}{N_L(t)} \\ &= \gamma^{\frac{\varepsilon}{\sigma}} \left(\frac{N_H(t)}{N_L(t)} \right)^{\frac{\sigma-1}{\sigma}} \left(\frac{H}{L} \right)^{-\frac{1}{\sigma}}. \end{aligned}$$

This expression shows why σ is often interpreted as the *derived* elasticity of substitution between the factors H and L: Holding the state variables fixed, σ describes how the equilibrium relative wage adjusts to changes in the factor input ratio:

$$\left. \frac{\partial \log(\omega(t))}{\partial \log(H/L)} \right|_{N_L(t), N_H(t)} = -\frac{1}{\sigma}.$$

The derived elasticity $\sigma = 1 + \beta(\varepsilon - 1)$ is naturally increasing in ε, so that the two factors are more substitutable (in general equilibrium) when their corresponding intermediates are more substitutable in final production. But notice that $\beta < 1$ "shrinks" the derived elasticity of substitution toward 1, because the machines are produced using final output and combined in a Cobb-Douglas fashion with each factors to produce the intermediates. More importantly, the two factors are "derived substitutes" if and only if their corresponding intermediates are substitutes in final production: $\sigma \gtrless 1 \iff \varepsilon \gtrless 1$.

This analysis completes our description of the static equilibrium in the model. We have written all "statically-determined" objects (prices, input quantities, and output quantities) as functions of exogenous parameters and the state variables $N_L(t)$ and $N_H(t)$.[2] Our next task is to use this characterization to determine how the state variables change dynamically in equilibrium.

[2]Recall that we can use the numeraire assumption and to write each of $p_L(t)$ and $p_H(t)$ as functions of $p(t)$: $p_L(t) = \left[\gamma_L^{\varepsilon} + \gamma_H^{\varepsilon} p(t)^{1-\varepsilon}\right]^{-\frac{1}{1-\varepsilon}}$.

2.2 Dynamic Equilibrium Conditions

I begin by studying the balanced growth path in this model, which features output and consumption growing at same constant rate g, a constant interest rate r, and a constant relative intermediate price p. For simplicity, I restrict attention to parameter values for which the growth rate is strictly positive, $g > 0$.

A positive growth rate requires that at least one of the sectors i has positive R&D expenditures, and it is easy to show that for the relative price p to remain constant *both* sectors must have positive R&D expenditures. As a result, the free-entry condition in each sector i requires that potential monopolists are exactly indifferent between investing final output into R&D:

$$\eta_i V_i(t) = 1.$$

But then $V_i(t) \equiv V_i$ is constant over time, so that the HJB equation reduces to

$$rV_i = \pi_i(t) = \bar{\pi} p_i(t)^{\frac{1}{\beta}} Q_i.$$

Since the relative price p is constant, the ideal price condition for the final good implies that $p_i(t) \equiv p_i$ must be constant. (Alternatively, just observe that this has to be true since the interest rate is constant on the BGP.) Using the previous two equations, we can derive the relation

$$r = \eta_i \bar{\pi} p_i^{\frac{1}{\beta}} Q_i.$$

Dividing the equation for H by the corresponding equation for L yields the BGP value for p:

$$p = \left(\eta \frac{H}{L}\right)^{-\beta},$$

where $\eta = \eta_H/\eta_L$. To relate p to the state variables $N_H(t)$ and $N_L(t)$, we recall that p must satisfy the following optimality condition from the final good producer's problem:

$$p = \gamma^{\frac{\beta\varepsilon}{\sigma}} \left(\frac{N_H(t)H}{N_L(t)L}\right)^{-\frac{\beta}{\sigma}}.$$

This implies that the ratio $N_H(t)/N_L(t) = (N_H/N_L)^*$ must be constant on the BGP. Substituting into the previous equation for p gives the BGP value

$$\left(\frac{N_H}{N_L}\right)^* = \eta^\sigma \gamma^\varepsilon \left(\frac{H}{L}\right)^{\sigma-1}.$$

This equation gives a complete characterization of the "directional" aspects of the BGP. To determine the growth rate g, we can make use of the household's Euler equation and the relation $r = \eta_L \bar{\pi} p_L^{\frac{1}{\beta}} L$:

$$\begin{aligned}
g &= \frac{1}{\theta}(r-\rho) \\
&= \frac{1}{\theta}\left(\eta_L \bar{\pi} p_L^{\frac{1}{\beta}} L - \rho\right) \\
&= \frac{1}{\theta}\left(\eta_L \bar{\pi}\left[\gamma_L^\varepsilon + \gamma_H^\varepsilon p^{1-\varepsilon}\right]^{-\frac{1}{\beta}\frac{1}{1-\varepsilon}} L - \rho\right) \\
&= \frac{1}{\theta}\left(\eta_L \bar{\pi}\left[\gamma_L^\varepsilon + \gamma_H^\varepsilon \left(\eta \frac{H}{L}\right)^{\sigma-1}\right]^{\frac{1}{\sigma-1}} L - \rho\right) \\
&= \frac{1}{\theta}\left(\bar{\pi}\left[\gamma_L^\varepsilon (\eta_L L)^{\sigma-1} + \gamma_H^\varepsilon (\eta_H H)^{\sigma-1}\right]^{\frac{1}{\sigma-1}} L - \rho\right).
\end{aligned}$$

This gives an essentially complete characterization of the BGP: We showed above that all statically-determined "relative" variables can be written as a function of N_H/N_L, and the common growth rate g of $N_H(t)$ and $N_L(t)$ determines the growth rate of all extensive variables (output, consumption, machine expenditures, R&D expenditures, $\ldots$). Finally, note that for this to be a valid BGP the growth rate must be strictly positive (our maintained assumption) and the household's transversality condition must be satisfied:

$$(1-\theta)\bar{\pi}\left[\gamma_L^\varepsilon (\eta_L L)^{\sigma-1} + \gamma_H^\varepsilon (\eta_H H)^{\sigma-1}\right]^{\frac{1}{\sigma-1}} L < \rho < \bar{\pi}\left[\gamma_L^\varepsilon (\eta_L L)^{\sigma-1} + \gamma_H^\varepsilon (\eta_H H)^{\sigma-1}\right]^{\frac{1}{\sigma-1}} L.$$

2.3 Equilibrium Bias of Technology

The most interesting features of this model concern its predictions for the factor bias of technology. Recall that, conditional on the state variables N_H and N_L, the relative wage ω is

$$\omega\left(\frac{H}{L}, \frac{N_H}{N_L}\right) = \gamma^{\frac{\varepsilon}{\sigma}} \left(\frac{N_H}{N_L}\right)^{\frac{\sigma-1}{\sigma}} \left(\frac{H}{L}\right)^{-\frac{1}{\sigma}}.$$

Here I make explicit the dependence of ω on the supply ratio H/L and the technology ratio N_H/N_L. We can observe that ω is always declining in the supply ratio, which is natural because $\omega\left(\frac{H}{L}, \frac{N_H}{N_L}\right)$ represents an inverse relative factor demand curve for fixed N_H/N_L. A change in the technology ratio N_H/N_L shifts this demand curve, and the direction is determined by σ: When $\sigma > 1$, an increase in N_H/N_L also increases $\omega\left(\frac{H}{L}, \frac{N_H}{N_L}\right)$ at each supply ratio H/L. In this case, a relatively H-*augmenting* technological change is also relatively H-*biased* because it raises the factor price ratio $\omega\left(\frac{H}{L}, \frac{N_H}{N_L}\right)$ at each supply ratio H/L. When instead $\sigma < 1$, an increase in N_H/N_L decreases $\omega\left(\frac{H}{L}, \frac{N_H}{N_L}\right)$, so that a relatively H-augmenting technological change is relatively L-biased.

These basic comparative statics treat technology as exogenously determined, but in this model technology is endogenous to the factor supplies. How does technology respond to changes in the supply ratio H/L on the BGP? Recall that the BGP technology ratio is

$$\left(\frac{N_H}{N_L}\right)^* = \eta^\sigma \gamma^\varepsilon \left(\frac{H}{L}\right)^{\sigma-1}.$$

Hence an increase in the supply ratio leads to relatively H-augmenting technological change if and only if $\sigma > 1$, with relatively L-augmenting technological change otherwise. Combining this observation with the relationship between factor-augmenting and factor-biased technological change discussed above, we recover the *weak equilibrium bias* result: An increase in H/L always leads to H-biased technological change. We can express this result mathematically as describing how ω responds to the change in N_H/N_L induced in equilibrium by the shift in H/L, holding fixed the direct effect of H/L on ω:

$$\frac{\partial \log(\omega)}{\partial \log(N_H/N_L)^*} \frac{d \log(N_H/N_L)^*}{d \log(H/L)} = \frac{(\sigma-1)^2}{\sigma} \geq 0.$$

This result implies that the endogenous response of technology to the relative factor supplies always counteracts the direct effect of H/L on the relative wage ω. The *strong equilibrium bias* result shows that when $\sigma > 2$ this response is so large as to imply an upward-sloping relative factor demand curve in general equilibrium:

$$\begin{aligned}
\frac{d \log(\omega)}{d \log(H/L)} &= \frac{\partial \log(\omega)}{\partial \log(H/L)} + \frac{\partial \log(\omega)}{\partial \log(N_H/N_L)^*} \frac{d \log(N_H/N_L)^*}{d \log(H/L)} \\
&= -\frac{1}{\sigma} + \frac{(\sigma-1)^2}{\sigma} \\
&= \sigma - 2.
\end{aligned}$$

This (surprising!) implication of the model suggests the importance of endogenizing the direction of technical change to understand the relationship between supply shocks and factor compensation in general equilibrium.