Innovation and Firm Dynamics

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These notes describe and solve a version of the Klette and Kortum (2004) model of innovation and firm dynamics.

1 Setup

The economy exists in continuous time and consists of a mass of research firms, a firm that produces final output, a measure $L > 0$ of workers, and a measure $S > 0$ of scientists. Each research firm produces intermediates and conducts research to improve their quality, and final output is produced by aggregating intermediates. Workers inelastically supply one unit of labor to research firms for the production of intermediates, and scientists inelastically supply one unit of research effort to research firms for the production of new innovations.

Final Production. At each time $t \in [0, \infty)$, final output is produced using the Cobb-Douglas production technology

$$Y(t) = \exp\left(\int_0^1 \log(x(\nu, t))\, d\nu\right). \tag{1.1}$$

Here $\nu \in [0, 1]$ indexes the intermediates used in final production and $x(\nu, t)$ denotes the quantity of intermediate ν used at time t. The corresponding price index (and the competitive price of final output) is

$$P(t) = \exp\left(\int_0^1 \log(p(\nu, t))\, d\nu\right), \tag{1.2}$$

where $p(\nu, t)$ is the price of intermediate ν at time t. The price of final output is normalized to one, $P(t) \equiv 1$, and the final output firm maximizes profits over input quantities:

$$\max_{x(\nu,t)\geq 0} \exp\left(\int_0^1 \log(x(\nu, t))\, dt\right) - \int_0^1 p(\nu, t)\, x(\nu, t)\, d\nu. \tag{1.3}$$

The solution yields the conditional intermediate demand functions

$$x(\nu, t) = \frac{Y(t)}{p(\nu, t)}. \tag{1.4}$$

Intermediate Production. Intermediate ν is produced by a research firm using only labor at marginal cost $w_L(t)/q(\nu, t)$, where $w_L(t)$ is the wage for labor and $q(\nu, t)$ denotes the *quality* of the intermediate. The quality of the intermediate increases over time through innovation, and we assume that each innovation on intermediate ν raises its quality by a factor $\lambda > 1$: If intermediate ν has quality $q(\nu, t)$ and a firm innovates at t, then the subsequent quality of this intermediate is

$$q(\nu, t+) = \lambda q(\nu, t). \tag{1.5}$$

A firm that innovates on intermediate ν receives a perpetual patent for the intermediate of the new quality $q(\nu, t+)$. Until a subsequent innovation, the firm engages in Bertrand-Nash competition with the intermediate of quality $\lambda^{-1} q(\nu, t+)$. Given that intermediate demand is unit elastic, at each time t intermediate prices satisfy the limit pricing condition

$$p(\nu, t) = \frac{w_L(t)}{\lambda^{-1} q(\nu, t)}. \tag{1.6}$$

The flow profits earned by the frontier variety of intermediate ν are then

$$\pi(\nu, t) = \left(p(\nu, t) - \frac{w_L(t)}{q(\nu, t)}\right) x(\nu, t) = \bar{\pi} Y(t), \tag{1.7}$$

where $\bar{\pi} = \frac{\lambda - 1}{\lambda}$.

Using the limit pricing condition (1.6), the price index for final output satisfies

$$1 = P(t) = \frac{\lambda w_L(t)}{Q(t)}, \tag{1.8}$$

where we define the aggregate quality of intermediates $Q(t)$ by

$$Q(t) = \exp\left(\int_0^1 \log(q(\nu, t))\, d\nu\right). \tag{1.9}$$

In particular, we note that the wage for labor can be written as a function of aggregate quality at each time t:

$$w_L(Q(t)) = \frac{Q(t)}{\lambda}. \tag{1.10}$$

Total labor used in the production of intermediates must also satisfy the market-clearing condition

$$L = \int_0^1 \frac{x(\nu,t)}{q(\nu,t)} d\nu = \frac{Y(t)}{\lambda w_L(t)}. \tag{1.11}$$

Final output at each time t can also be written as a function of aggregate quality:

$$Y(Q(t)) = Q(t)L. \tag{1.12}$$

The flow profits that accrue to the frontier variety of any intermediate are then

$$\pi(Q(t)) = \bar{\pi}Q(t)L. \tag{1.13}$$

Innovation. Innovation can be undertaken by incumbent and entrant firms through research. Considering first innovation by incumbents, we suppose that these firms can employ scientists to raise the rate at which they improve the quality of a random intermediate. Let $I \geq 0$ denote a firm's innovation rate, let $S \geq 0$ denote the firm's employment of research effort, and let $n \in \mathbb{N}$ denote the number of frontier intermediates produced by the firm. If $n \geq 1$, the innovation rate satisfies

$$I = G(S,n), \tag{1.14}$$

where the innovation production function G is strictly increasing in both arguments, strictly concave and smooth in S, and homogeneous of degree one. The corresponding "scientist requirement function" is

$$S = \Sigma(I,n) = n\sigma\left(\frac{I}{n}\right), \quad \text{where} \quad \sigma(\gamma) = \Sigma(\gamma,1). \tag{1.15}$$

We also assume that $\sigma(0) = 0$.

Consider an incumbent that produces $n \geq 1$ intermediates. Given a path of interest rates $r(t)$, the incumbent chooses its innovation rate $I(n,t)$ at each time t to maximize the expected present value of the firm $V(n,t)$, which satisfies the HJB equation

$$r(t)V(n,t) - \dot{V}(n,t) = \max_{I \geq 0} n\bar{\pi}Q(t)L - \Sigma(I,n)w_S(t) + I[V(n+1,t) - V(n,t)] \tag{1.16}$$

$$+ n\mu(t)[V(n-1,t) - V(n,t)]. \tag{1.17}$$

Here $w_S(t)$ denotes the wage rate for scientists at time t, and $\mu(t) \geq 0$ denotes the rate at which a random intermediate observes an innovation in equilibrium. Any incumbent that produces $n = 0$ intermediates permanently exits, so that $V(0,t) \equiv 0$. We restrict to equilibria in which the value function $V(n,t)$, the aggregate innovation rate $\mu(t)$, and the wage for research effort depend on time t only through aggregate quality $Q(t)$. In this case, we conjecture and verify that the value function takes the form

$$V(n,Q) = QLvn, \tag{1.18}$$

where $v > 0$ is a fixed constant. The boundary condition $V(0,t) = 0$ is clearly satisfied, and for $n \geq 1$ the HJB equation (1.16) becomes

$$r(t)Qv - \dot{Q}v = \max_{\gamma \geq 0} \bar{\pi}Q - \sigma(\gamma)\frac{w_S(Q)}{L} + (\gamma - \mu(Q))Qv. \tag{1.19}$$

The optimality condition for γ is

$$QLv = \sigma'(\gamma(Q))\,w_S(Q). \tag{1.20}$$

Now consider potential entrants. We assume that by employing $S^E \geq 0$ scientists, each entrant can attain innovation rate $\eta = \bar{\alpha}S^E$. Free entry implies the optimality conditions

$$\eta(Q) \geq 0 \quad \text{and} \quad QLv - \frac{w_S(Q)}{\bar{\alpha}} \leq 0, \tag{1.21}$$

with complementary slackness.

Consumption. The economy's representative consumer consumes final output, collects all earnings from labor and research effort, owns all research firms, and can purchase and sell a risk-free bond in zero net supply. Letting $C(t)$ denote final consumption at time t, standard arguments imply the Euler equation and transversality condition

$$\frac{\dot{C}(t)}{C(t)} = r(t) - \rho, \tag{1.22}$$

$$0 = \lim_{t\to\infty} \exp\left(-\int_0^t r(s)\,ds\right) Q(t), \tag{1.23}$$

where $\rho > 0$ is the consumer's discount rate and we assume that the consumer's constant elasticity of intertemporal substitution is equal to one. Market clearing requires $C(t) = Y(t) = Q(t)L$, so the Euler equation implies that the interest rate must satisfy

$$r(t) = \rho + \frac{\dot{Q}(t)}{Q(t)}. \tag{1.24}$$

The HJB equation (1.19) can then be written

$$\rho v = \max_{\gamma \geq 0} \bar{\pi} - \sigma(\gamma)\frac{w_S(Q)}{QL} + (\gamma - \mu(Q))\,v. \tag{1.25}$$

Factor Market Clearing. To describe the market-clearing equations for scientists, note that $\gamma(Q)$ and $\eta(Q)$ equal the aggregate innovation rates by incumbents and entrants, respectively. Market clearing for scientists requires

$$S = \sigma(\gamma(Q)) + \frac{\eta(Q)}{\bar{\alpha}}. \tag{1.26}$$

2 Equilibrium Characterization

To characterize the equilibria in this economy, first note that the incumbent and entry optimality conditions as well as the market-clearing constraint for scientists jointly imply that the wage for research effort must be linear in QL; we write $\bar{w}_S QL$ for the scientists' wage. As a result, γ and η must be constant in equilibrium, and the equilibrium conditions can be written

$$\text{HJB}: \quad v = \frac{\bar{\pi} - \sigma(\gamma)\bar{w}_S}{\rho + \eta}, \tag{2.1}$$

$$\text{Incumbent Optimality} \quad v = \sigma'(\gamma)\bar{w}_S, \tag{2.2}$$

$$\text{Entrant Optimality} \quad \frac{v}{\bar{w}_S} - \frac{1}{\bar{\alpha}} \leq 0 \quad \text{and} \quad \eta\left[\frac{v}{\bar{w}_S} - \frac{1}{\bar{\alpha}}\right] = 0, \tag{2.3}$$

$$\text{Market Clearing} \quad S = \sigma(\gamma) + \frac{\eta}{\bar{\alpha}}. \tag{2.4}$$

These equations imply a simple characterization of the unique equilibrium:

Proposition 1. *The equilibrium is unique and satisfies the following:*

(i) If $\sigma'\left(\sigma^{-1}(S)\right) \leq \frac{1}{\bar{\alpha}}$*, then*

$$\sigma(\gamma) = S, \quad \eta = 0, \quad \bar{w}_S = \frac{\bar{\pi}}{\rho\sigma'(\gamma) + \sigma(\gamma)}, \quad v = \frac{\bar{\pi}}{\rho + \frac{\sigma(\gamma)}{\sigma'(\gamma)}}. \tag{2.5}$$

(ii) If $\sigma'\left(\sigma^{-1}(S)\right) > \frac{1}{\bar{\alpha}}$*, then*

$$\sigma'(\gamma) = \frac{1}{\bar{\alpha}}, \quad \eta = \bar{\alpha}(S - \sigma(\gamma)), \quad \bar{w}_S = \frac{\bar{\pi}}{(\rho + \eta)\sigma'(\gamma) + \sigma(\gamma)}, \quad v = \frac{\bar{\pi}}{\rho + \eta + \frac{\sigma(\gamma)}{\sigma'(\gamma)}}. \tag{2.6}$$

Proof. Consider a candidate equilibrium with positive entry, $\eta > 0$. Then the entrant optimality condition (2.3) yields $\bar{w}_S = \bar{\alpha}v$, and the incumbent optimality condition determines the incumbent innovation rate:

$$\sigma'(\gamma) = \frac{1}{\bar{\alpha}}. \tag{2.7}$$

The market-clearing condition (2.4) requires $\gamma = \sigma^{-1}\left(S - \frac{\eta}{\bar{\alpha}}\right) < \sigma^{-1}(S)$, and since σ is strictly convex, the above equation can only hold when $\sigma'\left(\sigma^{-1}(S)\right) > \frac{1}{\bar{\alpha}}$. In this case, the market-clearing condition (2.4) yields $\eta = \bar{\alpha}(S - \sigma(\gamma))$, and the HJB and incumbent optimality conditions (2.1, 2.2) can be used to solve for $\bar{w}_S$ and v.

If $\sigma'\left(\sigma^{-1}(S)\right) \leq \frac{1}{\bar{\alpha}}$, then the argument above implies $\eta = 0$ and $\sigma(\gamma) = S$. Equations (2.1, 2.2) again determine $\bar{w}_S$ and v. □

The relative efficiency of the incumbent and entrant "innovation production functions" determines whether positive entry obtains in equilibrium. If $\bar{\alpha}$ is too low, entrants are insufficiently innovative, and all innovation is conducted by incumbents. The equilibrium growth rate is also simple to calculate in this economy:

Proposition 2. *If there is no entry in equilibrium, aggregate quality and output grow at rate*

$$\gamma \log(\lambda), \quad \textit{where} \quad \sigma(\gamma) = S. \tag{2.8}$$

With positive entry, aggregate quality and output grow at rate

$$[\gamma+\bar{\alpha}(S-\sigma(\gamma))]\log(\lambda), \quad \textit{where} \quad \sigma'(\gamma)=\frac{1}{\bar{\alpha}}. \tag{2.9}$$

The growth rate is increasing in the efficiency of entrant innovation $\bar{\alpha}$.

Proof. To prove the comparative statics result, note that with positive entry

$$\frac{d\gamma}{d\bar{\alpha}}=-\frac{1}{\bar{\alpha}^2\sigma''(\gamma)}, \tag{2.10}$$

$$\frac{d\bar{\alpha}(S-\sigma(\gamma))}{d\bar{\alpha}}=\frac{\sigma'(\gamma)}{\bar{\alpha}\sigma''(\gamma)}+S-\sigma(\gamma). \tag{2.11}$$

Hence

$$\frac{d[\gamma+\bar{\alpha}(S-\sigma(\gamma))]}{d\bar{\alpha}}=S-\sigma(\gamma)+\frac{\sigma'(\gamma)-1/\bar{\alpha}}{\bar{\alpha}\sigma''(\gamma)}=S-\sigma(\gamma). \tag{2.12}$$

□

For intuition about the comparative statics result, note that an increase in $\bar{\alpha}$ has two effects on incumbent and entrant innovation: It raises the efficiency of scientists currently employed by entrants and reallocates scientists away from incumbents and toward entrants in equilibrium. The reallocation effect has no first-order impact on aggregate growth, because free entry requires that the marginal scientist employed by an incumbent firm must be exactly as productive as the marginal scientist employed by an entrant. As a result, the economy can only grow faster as entrants innovate more rapidly after an improvement in their innovation efficiency $\bar{\alpha}$.