14.272 Recitation 1: Laffont-Tirole

Todd Lensman

February 14, 2025

Recitation Plan: Solve the canonical model of regulation under asymmetric information due to Laffont and Tirole (1993).[1]

1 Baseline Model

Consider the simplest possible setting of regulation or procurement: A government seeks to contract with one firm to carry out a project.[2] Completion of the project yields gross surplus $S > 0$ to consumers. The monetary cost of the project is $C = \beta - e$, where $\beta \in [\underline{\beta}, \bar{\beta}]$ is the baseline cost and $e \geq 0$ is the reduction in cost due to *effort* exerted by the firm. For example, this captures any planning or innovation that reduces the quantity of labor or materials needed to complete the project. Effort reduces the firm's utility by $\psi(e) \geq 0$ in money-metric terms, where ψ satisfies the regularity conditions

$$\psi(0) = 0, \qquad \lim_{e \uparrow \underline{\beta}} \psi(e) = \infty, \qquad \psi'(e), \psi''(e) > 0.$$

Note in particular that the marginal cost of effort increases as effort grows, $\psi''(e) > 0$.

The government can compensate the firm for its costs by paying a transfer. For accounting purposes, I suppose that the government always pays the cost C and the *net transfer* t. The firm's utility is then

$$U \equiv t - \psi(\underbrace{\beta - C}_{= e}).$$

To capture the social costs associated with distortionary taxation, I suppose that each dollar of public funds has a total resource cost of $1 + \lambda$ dollars, where $\lambda \geq 0$ denotes the *cost of public*

[1] See chapter 1 of Laffont and Tirole (1993) for *many* extensions of the basic model described in these notes.

[2] For example: building a bridge, designing a new missile, or digging underneath a prominent New England city to re-route two interstate highways.

funds. Utilitarian welfare (and the government's objective) is then

$$\begin{aligned} W &\equiv S-(1+\lambda)(C+t)+t-\psi(e) \\ &= S-(1+\lambda)(C+\psi(e))-\lambda U. \end{aligned}$$

When $\lambda = 0$, welfare is simply the gross surplus from the project S less its economic cost $C+\psi(e)$; the government is indifferent to the split of this net surplus between consumers and the firm. When $\lambda > 0$, collecting funds for the cost C and the transfer t is itself costly, so the government prefers to reduce the firm's utility (and hence the transfer t).

Throughout the analysis, I assume that the monetary cost C is always observable to the government. As a result, the government can design a potentially nonlinear schedule $t(C)$ describing the firm's net transfer as a function of the cost, which constitutes the contract offered to the firm. Given the contract $t(C)$, the firm chooses the cost (or equivalently its effort) to maximize its utility:

$$U_\beta \equiv \max_{C\in[0,\beta]} t(C)-\psi(\beta-C). \tag{1.1}$$

I will discuss both the "complete information" benchmark in which the baseline cost β and the firm's effort e are also observable to the government, as well as the more realistic case in which neither is observable.[3] In all cases, S is sufficiently large that the government always prefers to complete the project. As a result, the contract $t(C)$ must be such that the firm's equilibrium utility U_β is greater than its outside option, which is normalized to zero. This is known as the *individual rationality constraint*: $U_\beta \geq 0$ for all $\beta \in [\underline{\beta}, \bar{\beta}]$.

2 Complete Information

Suppose first that government can observe the baseline cost β and the firm's effort e. With no information asymmetry between the government and the firm, the government can design the contract $t(C)$ to achieve the *first-best* outcome, or the allocation that would obtain if the government had direct control of the firm. To see this, suppose that the government can directly

[3] Since C is observable, if one of β or e is observable then so is the other.

choose the transfer t and costs C with knowledge of β:

$$\max_{t,C} S-(1+\lambda)C-\psi(\beta-C)-\lambda t \tag{2.1}$$
$$\text{subject to}$$
$$t-\psi(\beta-C)\geq 0.$$

It is clearly optimal to set $t=\psi(\beta-C)$ to satisfy the individual rationality constraint while avoiding any extra cost of public funds. Assuming an interior choice of C, the necessary and sufficient first-order condition is then

$$\psi'(\beta-C)=1$$

With ψ strictly convex, this condition can be inverted to find the unique first-best cost C^*.

A variety of different contracts $t(C)$ can implement this first-best outcome. For example, the government could simply set

$$t(C)=\begin{cases}\psi(\beta-C^*) & \text{if } C=C^*,\\ -1 & \text{else.}\end{cases}$$

In this case, the firm realizes negative utility unless it chooses the first-best cost C^*, perhaps reflecting a threat by the government to shut down the firm. More realistically, the government could also adopt the *fixed-price contract*

$$t(C)=\psi(\beta-C^*)-(C-C^*).$$

Here the firm receives the lump-sum payment $C^*+\psi(\beta-C^*)>0$ and is the residual claimant on any cost-savings that result from its own effort. This is a *high-powered* contract that provides the firm the greatest incentives to achieve cost reductions through effort.[4]

Note a key feature of both of these contracts: The government must have perfect information about the baseline cost β. A more realistic assumption in many (if not all) settings is that the firm has more precise information about the cost of the project than the government. In the next section, I adopt a Bayesian version of this assumption and explore the implications for the government's optimal contract $t(C)$.

[4] Check for yourself that the solution to the firm's problem (1.1) under this contract is the first-best cost C^*!

3 Incomplete Information

Suppose now that the firm knows the baseline cost β, while the government only knows that β is distributed according to a smooth distribution F on $[\underline{\beta}, \bar{\beta}]$. Effort e is also unobservable. As before, the government must choose a potentially nonlinear contract $t(C)$ to maximize expected welfare $\mathbb{E}[W]$, taking as given utility-maximizing behavior by the firm (1.1) and the individual rationality constraint. The expectation is over the distribution F, capturing the assumption that the government does not know the baseline cost of the firm.

Transforming the Problem. The government's problem is exceptionally challenging as posed: Choosing an general nonlinear contract $t(C)$ to maximize expected welfare amounts to solving an infinite-dimensional optimization problem. To gain tractability, we can make use of an observation in the contract theory literature known as the *Revelation Principle* (Myerson, 1981):

Proposition 1 (Revelation Principle)**.** *For any contract $t(C)$, there exists a* direct mechanism $\{t_\beta, C_\beta\}_{\beta\in[\underline{\beta},\bar{\beta}]}$ *such that a firm of any type β chooses the same cost C_β and receives the same transfer t_β when confronted with the contract $t(C)$ as when confronted with the menu $\{t_\beta, C_\beta\}_{\beta\in[\underline{\beta},\bar{\beta}]}$. The direct mechanism satisfies the* incentive compatibility constraints

$$t_\beta - \psi(\beta - C_\beta) \geq t_{\beta'} - \psi(\beta - C_{\beta'}) \qquad \forall \beta, \beta' \in [\underline{\beta}, \bar{\beta}]. \tag{3.1}$$

The proof is immediate: For any schedule $t(C)$, we can simply forecast the cost C_β chosen by a firm with each baseline cost (or *type*) β, along with the resulting transfer $t_\beta = t(C_\beta)$. Optimality requires that the choice made by a firm of type β must dominate the choice made by a firm of any other type β', yielding the incentive compatibility conditions (3.1). So we can equivalently formulate the government's problem as an optimization problem over direct mechanisms $\{t_\beta, C_\beta\}_\beta$ that satisfy the incentive compatibility constraints (3.1). As we will see below, under mild technical conditions any such direct mechanism is equivalent to the choice of a nonlinear contract $t(C)$. Thus the government's problem is equivalent when formulated as the choice of a contract or as the choice of an incentive-compatible direct mechanism.[5]

How do direct mechanisms aid tractability? When the distribution F is discrete, so that there are only finitely many types β, the Revelation Principle offers a substantial simplification. Instead of choosing an infinite-dimensional contract $t(C)$, the government can instead choose a

[5]This second step, moving from a direct mechanism back to a nonlinear contract, is called an *implementation problem*.

finite-dimensional direct mechanism $\{t_\beta, C_\beta\}_\beta$ subject to the incentive compatibility constraints (3.1). But I assume that F is smooth, so the government's problem is still infinite-dimensional. To make progress, we will assume that the direct mechanism $\{t_\beta, C_\beta\}_\beta$ is twice continuously differentiable in β, so that we can make use of first-order conditions to characterize the solution to the firm's problem.[6]

Under a direct mechanism $\{t_\beta, C_\beta\}_\beta$, a firm of type β chooses a type $\hat{\beta}$ to report, exerts effort to produce the corresponding cost $C_{\hat{\beta}}$, and receives the corresponding transfer $t_{\hat{\beta}}$. The firm's problem (1.1) then becomes

$$U_\beta = \max_{\hat{\beta} \in [\underline{\beta}, \bar{\beta}]} t_{\hat{\beta}} - \psi(\beta - C_{\hat{\beta}}). \tag{3.2}$$

Given incentive compatibility, the optimal choice must be $\hat{\beta} = \beta$. The corresponding first-order condition is

$$\dot{t}_\beta = -\psi'(\beta - C_\beta)\dot{C}_\beta, \tag{3.3}$$

where I use the $\dot{}$ notation to denote derivatives with respect to the report $\hat{\beta}$. In the appendix, I show that the incentive compatibility conditions (3.1) imply that $\dot{C}_\beta \geq 0$, and that under this restriction the first-order condition (3.3) is necessary and sufficient to characterize the solution to the firm's problem (3.2). As a result, we can replace the incentive-compatibility conditions (3.1) with the first-order condition (3.3) and the monotonicity condition $\dot{C}_\beta \geq 0$.

For a final simplification, it will be convenient to reformulate the choice of a direct mechanism $\{t_\beta, C_\beta\}_\beta$ instead as the choice of $\{U_\beta, C_\beta\}_\beta$, where the transfer t_β is implicitly defined by $U_\beta = t_\beta - \psi(\beta - C_\beta)$.[7] In this case, the first-order condition (3.3) is equivalent to the envelope condition

$$\dot{U}_\beta = -\psi'(\beta - C_\beta). \tag{3.4}$$

Note that this condition immediately implies that U_β is decreasing, so the individual rationality condition $U_\beta \geq 0$ binds only for the firm with the highest baseline cost $\bar{\beta}$. Given these

[6] All of what follows actually holds under weaker regularity conditions, at the cost of more technically challenging arguments.

[7] This is known as the "Mirrlees trick" after Mirrlees (1971).

observations, we can reformulate the government's problem as follows:

$$\max_{\{U_\beta, C_\beta\}_\beta} \int_{\underline{\beta}}^{\bar{\beta}} \left[S - (1+\lambda)(C_\beta + \psi(\beta - C_\beta)) - \lambda U_\beta \right] f(\beta) d\beta \tag{3.5}$$

subject to

$$\dot{U}_\beta = -\psi'(\beta - C_\beta),$$

$$\dot{C}_\beta \geq 0,$$

$$U_{\bar{\beta}} \geq 0.$$

As stated, this is a finite-horizon optimal control problem with "time" variable β, state variable U_β, and control variable C_β. The envelope condition (3.4) is the evolution equation for the state variable U_β. The condition $\dot{C}_\beta \geq 0$ is a pointwise constraint on the control variable.[8] The condition $U_{\bar{\beta}} \geq 0$ is a constraint on the terminal value of the state variable, which binds at the optimum.

Solution. The solution to the government's problem (3.5) can be characterized using the Maximum Principle. For a simpler approach, we can go through a bit of algebra to drop U_β from the problem. Note first that the Fundamental Theorem of Calculus implies

$$U_\beta = U_{\bar{\beta}} + \int_\beta^{\bar{\beta}} \psi'(\tilde{\beta} - C_{\tilde{\beta}}) d\tilde{\beta}.$$

Individual rationality requires that $U_{\bar{\beta}} = 0$ at the optimum, so the expected rent given up by the government equals

$$\begin{aligned}
\int_{\underline{\beta}}^{\bar{\beta}} U_\beta f(\beta) d\beta &= \int_{\underline{\beta}}^{\bar{\beta}} \int_\beta^{\bar{\beta}} \psi'(\tilde{\beta} - C_{\tilde{\beta}}) f(\beta) d\tilde{\beta} d\beta \\
&= \int_{\underline{\beta}}^{\bar{\beta}} \int_{\underline{\beta}}^{\tilde{\beta}} \psi'(\tilde{\beta} - C_{\tilde{\beta}}) f(\beta) d\beta d\tilde{\beta} \\
&= \int_{\underline{\beta}}^{\bar{\beta}} \psi'(\beta - C_\beta) F(\beta) d\beta.
\end{aligned}$$

[8] This is really a pointwise constraint on the *derivative* of the control variable, which must be handled differently.

The second line follows by interchanging the order of integration. Substituting back into the government's objective yields

$$\int_{\underline{\beta}}^{\bar{\beta}} \left[S - (1+\lambda)(C_\beta + \psi(\beta - C_\beta)) - \lambda \frac{F(\beta)}{f(\beta)} \psi'(\beta - C_\beta) \right] f(\beta) d\beta. \tag{3.6}$$

We can solve the government's problem (3.5) by maximizing this objective with respect to C_β, subject to the pointwise constraint $\dot{C}_\beta \geq 0$. Standard practice is to conjecture and verify that this constraint is non-binding. If this holds, we can simply maximize the objective (3.6) pointwise with no constraints on C_β. The optimal choice of C_β then satisfies the first-order condition

$$0 = -(1+\lambda)(1 - \psi'(\beta - C_\beta)) f(\beta) + \lambda F(\beta) \psi''(\beta - C_\beta). \tag{3.7}$$

Intuitively, increasing the cost C_β reduces welfare directly (first term). But it also increases welfare indirectly by reducing the information rents that must be paid to types with lower baseline costs (second term). Provided that $\psi'''(e) \geq 0$, this condition is both necessary and sufficient to determine the optimal cost C_β.

To verify that the cost function C_β is weakly increasing, we can implicitly differentiate the first-order condition (3.7) to find

$$1 - \dot{C}_\beta = -\frac{\lambda \psi''(\beta - C_\beta) \frac{d}{d\beta}\left(\frac{F(\beta)}{f(\beta)}\right)}{(1+\lambda)\psi''(\beta - C_\beta) + \lambda \frac{F(\beta)}{f(\beta)} \psi'''(\beta - C_\beta)}$$

With $\psi'''(e) \geq 0$, we observe that a *sufficient condition* for monotonicity is for the distribution F to satisfy the hazard rate condition:

$$\frac{d}{d\beta} \frac{F(\beta)}{f(\beta)} \geq 0.$$

This condition is violated when $f(\beta)$ is increasing relative to $F(\beta)$ as β increases. In line with the interpretation of the first-order condition (3.7), this would push the government to *reduce* costs C_β as β increases, because the direct effect of lower costs dominates the indirect effect of greater information rents for more efficient types. In fact, under this condition the equilibrium effort exerted by the firm $\beta - C_\beta$ is *decreasing* with β: A higher β not only increases the baseline cost, but it also lowers the cost savings from effort.

The following proposition summarizes the solution to the government's problem:

Proposition 2. *Suppose $\psi'''(e) \geq 0$ and $F(\beta)/f(\beta)$ is weakly. Then the government's optimal direct mechanism $\{t_\beta, C_\beta\}_\beta$ uniquely satisfies*

$$
\begin{aligned}
0 &= -(1+\lambda)(1-\psi'(\beta - C_\beta))f(\beta) + \lambda F(\beta)\psi''(\beta - C_\beta),\\
U_\beta &= \int_\beta^{\bar{\beta}} \psi'(\tilde{\beta} - C_{\tilde{\beta}})d\tilde{\beta},\\
t_\beta &= U_\beta + \psi(\beta - C_\beta).
\end{aligned}
$$

Lessons for Regulation. The analysis above delivers a few concrete lessons for optimal regulation. To see this, note first that we can define a nonlinear contract $t(C)$ to implement the optimal direct mechanism by setting

$$t(C_\beta) \equiv t_\beta \qquad \forall \beta \in [\underline{\beta}, \bar{\beta}]. \tag{3.8}$$

The contract $t(C)$ is well-defined under the assumptions of Proposition 2, which ensure that C_β is strictly increasing in β.[9] The shape of the contract $t(C)$ determines the incentives facing the firm, because it controls the firm's private return to effort. Note first that the net transfer paid to firm with the highest baseline cost is positive:

$$t(C_{\bar{\beta}}) = \psi(\bar{\beta} - C_{\bar{\beta}}).$$

This equation follows directly from Proposition 2. It holds because the individual rationality constraint must bind for the highest-cost firm, so that this firm is just compensated for its effort. By implicitly differentiating (3.8), we see that the net transfer increases as costs fall:

$$t'(C_\beta) = \frac{\dot{t}_\beta}{\dot{C}_\beta} = -\psi'(\beta - C_\beta).$$

This increase goes beyond compensating a firm with lower baseline cost for its higher effort, because Proposition 2 shows that each such firm earns a positive *information rent* $U_\beta > 0$. A key lesson from the analysis is that allowing some information rents to more efficient types is necessary to incentivize them to exert effort; the size of these rents (and hence the effort exerted) is limited because public funds are socially costly. For the firm with the lowest baseline

[9] We can set t negative outside of the image of C_β to define t everywhere.

cost, the first-order condition (3.7) implies that the firm faces first-best effort incentives:

$$1 = \psi'(\underline{\beta} - C_{\underline{\beta}}).$$

This holds because the government need not provide information rents to any firm with a lower baseline cost. Finally, by differentiating $t'(C_\beta)$ again we find that the optimal contract is convex, so that the firm is rewarded with a larger transfer at an increasing rate as its realized cost C falls:

$$t''(C_\beta) = -\psi''(\beta - C_\beta)\left(\frac{1}{\dot{C}_\beta} - 1\right).$$

Hence the contract t resembles a fixed-price contract for low costs C, smoothly reducing the power of incentives to move closer to a cost-plus contract as costs increase.

Intuition. For a different intuition about the shape of the optimal contract $t(C)$, suppose we began by offering the firm a fixed-price contract of the firm $t(C) = a - C$. This contract makes the firm the residual claimaint on any cost reductions achieved through effort, so the firm always exerts the first-best amount of effort. To ensure individual rationality for all types, we must choose a sufficiently large that the firm with the highest baseline cost $\bar{\beta}$ attains zero utility. But since this lump sum must also be delivered to more efficient types, it comes at a relatively high (expected) cost of public funds: We could improve the contract by reducing the lump-sum payment a but raising the transfer at high values of the cost C. This sacrifices some incentives for cost reduction for inefficient types (who already choose high costs C) to save on the social cost of a high lump-sum payment for all types. This adjustment lowers and "flattens" the contract $t(C)$, producing the shape described by Proposition 2. The optimal contract $t(C)$ is "more convex" for higher values of λ; it is exactly linear in the limit with no social cost of public funds, $\lambda = 0$.

References

Laffont, J.-J., & Tirole, J. (1993). *A theory of incentives in procurement and regulation*. MIT Press.

Mirrlees, J. A. (1971). An exploration in the theory of optimum income taxation. *The review of economic studies*, *38*(2), 175–208.

Myerson, R. B. (1981). Optimal auction design. *Mathematics of operations research*, *6*(1), 58–73.

A Technical Details

A.1 Monotonicity of Costs

Monotonicity of the costs C_β in β arises twice in the analysis: First, incentive compatibility implies that C_β is weakly increasing. Second, provided that C_β is weakly increasing, the first-order condition is both necessary and sufficient to characterize the solution to the firm's problem under a direct mechanism. Here I prove both of these facts.

IC ⇒ Monotonicity. The incentive compatibility conditions for types β and $\beta' > \beta$ are

$$t_\beta - \psi(\beta - C_\beta) \geq t_{\beta'} - \psi(\beta - C_{\beta'}),$$
$$t_{\beta'} - \psi(\beta' - C_{\beta'}) \geq t_\beta - \psi(\beta' - C_\beta).$$

Adding these two inequalities yields

$$\psi(\beta' - C_\beta) - \psi(\beta - C_\beta) \geq \psi(\beta' - C_{\beta'}) - \psi(\beta - C_{\beta'}).$$

Since ψ is strictly convex, this inequality holds if and only if $C_{\beta'} \geq C_\beta$. Hence $\dot{C}_\beta \geq 0$.

Monotonicity ⇒ FOC Sufficiency. Fix types β and $\beta' > \beta$. Note the difference

$$\begin{aligned} t_{\beta'} - \psi(\beta' - C_{\beta'}) - \left[t_\beta - \psi(\beta' - C_\beta)\right] &= \int_\beta^{\beta'} \left[\dot{t}_{\tilde{\beta}} + \psi'(\beta' - C_{\tilde{\beta}})\dot{C}_{\tilde{\beta}}\right] d\tilde{\beta} \\ &= \int_\beta^{\beta'} \left[\psi'(\beta' - C_{\tilde{\beta}}) - \psi'(\tilde{\beta} - C_{\tilde{\beta}})\right] \dot{C}_{\tilde{\beta}} \, d\tilde{\beta} \\ &\geq 0. \end{aligned}$$

The first line holds by the Fundamental Theorem of Calculus, while the second line follows from the first-order condition (3.3). The last line holds by the convexity of ψ, which implies that the bracketed term in the integrand is positive, and the assumption that C_β is weakly increasing.

A.2 Monotonicity of C_β

To verify that the cost function C_β is weakly increasing, we implicitly differentiate the first-order condition (3.7):

$$0 = -(1+\lambda)(-\psi''(\beta - C_\beta)\dot{C}_\beta)f(\beta) + \lambda\left(f(\beta)\psi''(\beta - C_\beta) + F(\beta)\psi'''(\beta - C_\beta)\dot{C}_\beta\right).$$

Simplifying, we get:

$$0 = (1+\lambda)\psi''(\beta - C_\beta)f(\beta)\dot{C}_\beta - \lambda F(\beta)\psi'''(\beta - C_\beta)\dot{C}_\beta.$$

Rearranging, we obtain:

$$\dot{C}_\beta = \frac{(1+\lambda)\psi''(\beta - C_\beta)f(\beta)}{\lambda F(\beta)\psi'''(\beta - C_\beta)}.$$

Since $\psi'''(e) \geq 0$ and $F(\beta)/f(\beta)$ is weakly increasing, $\dot{C}_\beta \geq 0$.