14.272 Recitation 3a: The Cournot Model

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Recitation Plan: Solve the baseline Cournot model, describe regularity conditions, and explore key economic implications.[1]

1 Baseline Model

Consider a market for a homogeneous good with a mass of consumers and a finite number of firms $i \in \{1, \ldots, N\}$.[2]

Consumers. Consumers derive gross surplus $S(X)$ from consuming an aggregate quantity X of the good, generating the inverse demand function

$$P(X) = S'(X). \tag{1}$$

The surplus function $S(\cdot)$ is assumed continuous, strictly increasing, and strictly concave, so that the inverse demand function $P(\cdot)$ is continuous, positive, and strictly decreasing.

Firms. Firm i can produce quantity $x_i \geq 0$ of the good at total cost $c_i(x_i)$, assumed weakly increasing. Firms compete in quantities to maximize profits with Nash conjectures. Given $X_{-i} \equiv \sum_{j \neq i} x_j$, the aggregate quantity produced by all other firms, firm i then solves

$$\max_{x_i \geq 0} \underbrace{P(x_i + X_{-i})x_i - c_i(x_i)}_{\pi_i(x_i, X_{-i})}. \tag{2}$$

Let $b_i(X_{-i})$ denote the solution set to this problem, or firm i's *best-response correspondence*.

Equilibrium. An *equilibrium* is a vector of quantities $(x_1^*, \ldots, x_N^*)$ such that x_i^* solves the firm i's problem (2) with $X_{-i} = \sum_{j \neq i} x_j^*$.

[1] These notes build on ones by Mike Whinston.

[2] See, for example, recent empirical applications to the hard-drive disk industry (Igami and Uetake, 2020) and the oil industry (Asker et al., 2024)

Remark. Interpreted literally, the Cournot model requires each firm to forecast the equilibrium price under any choice of its quantity, without specifying *who actually sets the price*. Just as in a competitive economy, the price is simply assumed to balance demand and supply. This (lack of an) explanation for how the price is actually set is strange – especially for a model meant to describe competition across a small number of firms! Kreps and Scheinkman (1983) offer a very elegant alternative interpretation: Consider a two-stage game in which each firm first builds production capacity x_i at cost $c_i(x_i)$. In the second stage, each firm sets a price p_i and supplies output to consumers up to its pre-committed capacity (or the quantity demanded). In equilibrium, all firms choose the same price, this price equates total demand to total supply, and firm quantities correspond to the Cournot equilibrium outcome.

2 Characterizing Equilibrium

To characterize the equilibrium, we begin by studying the firm's problem (2). The *necessary* first-order condition is

$$0 \leq \frac{\partial \pi_i(x_i^*, X_{-i}^*)}{\partial x_i} = P'(X^*)x_i^* + P(X^*) - c_i'(x_i^*) \quad \text{and} \quad x_i^* \geq 0, \tag{3}$$

with complementary slackness. Throughout I write $X = \sum_j x_j$ for the aggregate quantity. The first-order condition (3) has a natural interpretation: Firm i raises its quantity x_i until the marginal benefit from selling an additional unit at price $P(X)$ balances the marginal cost $c_i'(x_i)$ and the revenues lost on all x_i inframarginal units because of a decline in the price $P'(X)$.

In general, the first-order condition (3) is not sufficient to characterize solutions to the firm's problem: The profit function $\pi_i(x_i, X_{-i})$ may fail to be quasi-concave in x_i for arbitrary inverse demand functions $P(\cdot)$ and when cost functions $c_i(\cdot)$ feature increasing returns to scale. Equilibria may then fail to exist or exhibit counterintuitive comparative statics. To rule out these pathologies, it is standard to assume the following (e.g., Farrell and Shapiro, 1990):

$$P''(X)X + P'(X) \leq 0 \quad \text{for all } X \geq 0 \text{ such that } P(X) \geq 0 \tag{A1}$$

$$P'(X) - c_i''(x_i) < 0 \quad \text{for all } x_i \in [0, X] \text{ and } X \geq 0 \text{ such that } P(X) \geq 0 \tag{A2}$$

Note, for example, that (A1) is implied by the assumption that the indirect demand function $P(\cdot)$ is weakly concave, which may not always be reasonable.[3] Similarly, (A2) is implied by the assumption that all cost functions feature non-decreasing marginal costs $c_i''(\cdot) \geq 0$.

[3] For example, (A1) is violated when aggregate demand features a constant elasticity: $P(X) \propto X^{-\frac{1}{\varepsilon}}$. In such

To see the key implications of (A1, A2), suppose that the cost functions $c_i(\cdot)$ are continuous, ruling out fixed costs, and that $\lim_{x_i \to \infty} \pi_i(x_i, 0) < -c_i(0)$, ensuring a finite aggregate quantity X. Then a solution to the firm's problem (2) must exist. The assumptions (A1, A2) together imply that the profit function $\pi_i(x_i, X_{-i})$ is strictly concave in x_i, so that the solution is unique. Moreover, they imply that $\pi_i(x_i, X_{-i})$ has increasing differences in $(-x_i, X_{-i})$:

$$\frac{\partial^2 \pi_i(x_i, X_{-i})}{\partial X_{-i} \partial x_i} \;=\; P''(X)x_i + P'(X) \;\leq\; 0.$$

Hence the best-response function $b_i(X_{-i})$ must be weakly decreasing, and strictly so whenever the previous inequality is strict at $x_i = b_i(X_{-i})$ with $b_i(X_{-i}) > 0$. Moreover, whenever $b_i(X_{-i}) > 0$, we can differentiate the interior first-order condition (3) to find

$$b_i'(X_{-i}) \;=\; -\frac{P''(X)x_i + P'(X)}{P''(X)x_i + 2P'(X) - c_i''(x_i)} \in (-1, 0]. \tag{4}$$

The lower bound $b_i'(X_{-i}) > -1$ follows from (A1, A2). This expression clarifies two key features of the Cournot model:

1. **Strategic substitutes:** With $b_i'(X_{-i}) \leq 0$, each firm responds in the opposite direction to any adjustments made by other firms. An increase in the quantity X_{-i} supplied by all firms other than i will be met with a decrease in the quantity x_i supplied by firm i.

2. **Stability:** With $b_i'(X_{-i}) > -1$, each firm responds *less than one-for-one* to any adjustments made by other firms. This ensures that best response dynamics are *stable* in the sense that a perturbation to one firm's quantity (say, a response to a small change in marginal costs) will not generate drastic changes in the equilibrium aggregate quantity X^*.

Finally, to characterize the model's equilibria, it is helpful to define the *fitting-in function* $\phi_i(X)$ as the solution to

$$\phi_i(X) = b_i(X - \phi_i(X)).$$

Given an aggregate quantity X, $\phi_i(X)$ gives the quantity for firm i consistent with both optimality in the firm's problem (2) and the requirement that the aggregate quantity should be X.

cases, equilibrium analysis may be facilitated by a weaker version of (A1):

$$P''(X)x_i + P'(X) \leq 0 \quad \text{for all } x_i \in (0, X] \text{ and } X \geq 0 \text{ satisfying (3)}.$$

The assumption (A1) also imposes "Marshall's Second Law of Demand": The demand elasticity $\varepsilon(P(X)) \equiv -\frac{P(X)}{P'(X)X}$ is non-increasing in X.

Note that $\phi_i(\cdot)$ is well-defined since $b_i'(\cdot) > -1$. An equilibrium aggregate quantity X^* must then satisfy the fixed-point equation

$$X^* = \sum_i \phi_i(X^*).$$

It follows from (4) that each fitting-in function $\phi_i(\cdot)$ is non-increasing, so this equation has at most one solution. Given that each firm's equilibrium quantity is determined by $x_i^* = \phi_i(X^*)$, this implies that the equilibrium must be unique.[4] Note this *extremely* special and useful property: Each firm i interacts strategically with all other firms only through the aggregate quantity X. The existence of a one-dimensional index that captures all strategic interactions makes the Cournot model the classic example of an *aggregative game*, and it is responsible for a number of convenient features of the model. See Jensen (2010) and Acemoglu and Jensen (2013) for analyses that generalize many features of the Cournot model to the basic setting of an aggregative game with strategic substitutes.

3 Economic Implications

The Cournot model provides several baseline intuitions about the economics of imperfect competition. To draw these out, I assume an interior equilibrium:

$$0 \;=\; P'(X^*)x_i^* + P(X^*) - c_i'(x_i^*) \quad \text{for} \quad i \in \{1, \ldots, N\}. \tag{5}$$

1. Allocative Inefficiency. The equilibrium conditions (5) immediately imply that firms price above marginal cost, $P(X^*) > c_i'(x_i)$. This is inefficient in a partial equilibrium economy.[5]

2. Comparative Statics with Symmetry. The Cournot model delivers particularly intuitive comparative statics when all firms are symmetric, $c_i(x_i) = c(x_i)$. In this case, the equilibrium conditions (5) reduce to the single equation

$$0 \;=\; P'(X^*)\frac{X^*}{N} + P(X^*) - c'\left(\frac{X^*}{N}\right). \tag{6}$$

[4] Under the continuity assumptions and (A1, A2), the Debreu-Glicksberg-Fan Theorem ensures existence.

[5] Not necessarily true in general equilibrium: As far back as Samuelson (1947), it was recognized that *heterogeneous* markups are needed for inefficiency when factor prices adjust endogenously.

Equivalently, we can use the symmetric best-response function $b(\cdot)$ to write

$$X^* = N\,b\left(\frac{N-1}{N}X^*\right).$$

Implicitly differentiating with respect to N and using the lower bound $b'(\cdot) > -1$ implies that X^* is strictly increasing in N. With the assumption $\lim_{x\to\infty}\pi(x,0) < -c(0)$, we also have that X^* limits to a finite upper bound $\bar{X}$ as $N \to \infty$. The symmetric equilibrium condition (6) then implies that the equilibrium smoothly interpolates between the monopoly outcome and the perfectly competitive outcome as N increases from 1 to ∞.

3. Production Inefficiency. Consider two firms i and j with $x_i^*, x_j^* > 0$. Subtracting the corresponding first-order conditions (5) yields

$$c_i'(x_i^*) - c_j'(x_j^*) \;=\; P'(X^*)[x_i^* - x_j^*]. \tag{7}$$

We immediately observe the following implications:

- Larger firms produce too little. If $x_i^* > x_j^*$, then (7) implies $c_i'(x_i^*) < c_j'(x_j^*)$. Aggregate welfare could be increased by reallocating production from j to i, holding fixed the aggregate quantity.

- Efficient firms produce too little. Suppose all firms have non-decreasing marginal costs, and suppose that firm i is uniformly more efficient than firm j: $c_i'(x) < c_j'(x)$ for all $x \geq 0$. Then we must have $x_i^* > x_j^*$. Otherwise, greater efficiency by i and non-decreasing marginal costs would imply $c_i'(x_i^*) < c_j'(x_j^*)$, contradicting the above equation since $P'(\cdot) < 0$. Aggregate welfare could again be increased by reallocating production from j to i.

- Larger or more efficient firms have greater margins, Lerner indices, and markups. This follows immediately from the observations above. But note then that in the cross-section of firms, *margins and markups reflect production efficiency in addition to market power*!

4. Lerner Indices and Market Shares. Letting $\alpha_i^* \equiv x_i^*/X_i^*$ denote firm i's market share, we can rearrange the equilibrium conditions (5) to write firm i's Lerner index in terms of the local elasticity of demand $\varepsilon(P(X^*)) \equiv -\frac{P(X^*)}{P'(X^*)X^*}$ and α_i^*:

$$\mathcal{L}_i^* \;\equiv\; \frac{P(X^*) - c_i'(x_i^*)}{P(X^*)} \;=\; \frac{\alpha_i^*}{\varepsilon(P(X^*))}.$$

We again observe that firms with larger market shares have larger Lerner indices. This equation also implies a simple relationship between the share-weighted average Lerner index in the market and the Herfindahl-Hirschman index (HHI):

$$\mathcal{L}^* \equiv \sum_i \mathcal{L}_i^* \alpha_i^* = \frac{\text{HHI}^*}{\varepsilon(P(X^*))}.$$

5. Prices and Concentration. The Cournot model immediately shows that we should expect no clear empirical relationship between prices and concentration measures like HHI. For example, we saw above in the symmetric case that price is declining in the number of firms N, suggesting an increasing relationship between price and concentration. But if we start with symmetric firms and increase one firm i's marginal cost slightly, this will both increase price and concentration: Firm i reduces its quantity, which is only partially offset in equilibrium by quantity expansions by the other firms. Hence the aggregate quantity falls, increasing the price, and firm i loses market share to all other firms, raising HHI.

6. Welfare and Concentration. Just as the Cournot model offers no clear prediction for the relationship between the market price and concentration, the relationship between aggregate welfare and concentration is ambiguous. For example, in the symmetric case with constant marginal costs, welfare is clearly smaller with a monopolist ($N = 1$) than in the competitive limit ($N \to \infty$). But starting from an equilibrium with asymmetric firms, suppose we made small adjustments dx_i to each firm i's quantity, holding the aggregate quantity fixed:

$$\sum_i dx_i = 0.$$

The change in welfare is then

$$dW = -\sum_i c_i'(x_i^*)dx_i = -P'(X^*)\sum_i x_i^* dx_i.$$

The second equality holds after substituting the equilibrium conditions (5). But we can also differentiate the HHI to find

$$d\text{HHI} = \frac{2}{X^{*2}}\sum_i x_i^* dx_i.$$

Substituting into the previous equation yields

$$dW = -P'(X^*)X^{*2}\frac{d\text{HHI}}{2}.$$

Hence welfare *increases* provided that the quantity reallocation increases HHI! The intuition follows from the discussion above on production inefficiency: Larger firms have lower marginal costs than smaller firms, and reallocating quantity toward these larger firms increases both concentration and aggregate welfare.

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