14.272 Recitation 3b: Nocke-Schutz Oligopoly

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Recitation Plan: Solve the model of multi-product firm oligopoly with aggregative demand due to Nocke and Schutz (2018), with an application to horizontal merger analysis.[1]

Motivation: Models of imperfect competition with general demand systems are often impossible to solve analytically, and it is can be very difficult to characterize comparative statics with respect to primitives like marginal costs. These difficulties are further compounded when firms can own and price multiple imperfectly substitutable products – a key feature of any horizontal merger analysis, excluding markets for homogeneous goods.[2] In a series of papers, Nocke and Schutz (2018, 2025) make progress on these issues by describing a class of demand systems that imply the corresponding pricing game is *aggregative*, lending substantial analytical tractability. These demand systems satisfy IIA and can be microfounded through a discrete-continuous choice problem; Nocke and Schutz (2025) explore extensions to demand systems that do not satisfy IIA, including nested logit and nested CES.

1 Setup

Consider a market for a finite set $\mathcal{N}$ of imperfectly substitutable goods. The market has a mass of consumers and a finite set $\mathcal{F}$ of firms, each of which is identified with the set of goods it owns: $f \subseteq \mathcal{N}$ for $f \in \mathcal{F}$.

Demand. The representative consumer has quasi-linear preferences over the goods $\mathcal{N}$. These preferences are defined through an indirect utility function: Given a price vector p, consumer surplus is given by $V(p) = \log(H(p))$, where the *aggregator* $H(p)$ is defined by

$$H(p) \equiv \sum_{i \in \mathcal{N}} h_i(p_i) + H^0. \tag{1}$$

[1] These notes build on ones by Thi Mai Anh Nguyen.

[2] The classic analysis of Deneckere and Davidson (1985) preserves tractability by assuming a linear demand system, but such a strong assumption sheds doubt on the robustness of their results.

Here the functions $h_i(\cdot)$ are assumed smooth, strictly positive, strictly decreasing, and log-convex, while $H^0 \geq 0$ is a constant. These technical conditions are needed to ensure that $V(\cdot)$ defines a proper indirect utility function. Each function $\log(h_i(\cdot))$ can be interpreted as an indirect utility function in a discrete-continuous choice problem where each consumer first chooses a good to consume and then chooses its quantity; $\log(H^0)$ is then the indirect utility of the outside option.

Roy's Identity (i.e., the Envelope Theorem) implies that demand for good $i \in \mathcal{N}$ is

$$D_i(p) \quad = \quad -\frac{\partial V(p)}{\partial p_i} \quad = \quad -\frac{h_i'(p_i)}{H(p)} \quad \equiv \quad \hat{D}_i(p_i, H(p)).$$

The notation $\hat{D}_i(p_i, H(p))$ emphasizes the key assumption embedded in the demand system: The demand for each product i depends on the price of every other good *only through the aggregator* $H(p)$. This is a kind of symmetry assumption that places considerable discipline on substitution patterns. For example, given the functional forms assumed here, cross-price elasticities depend only on the good whose price is changing:

$$\frac{\partial \log(D_i(p))}{\partial \log(p_k)} = -\frac{\partial \log(H(p))}{\partial \log(p_k)}.$$

The demand system also satisfies the Independence of Irrelevant Alternatives (IIA) property: The ratio of demands D_i/D_j is independent of the prices of all other goods $k \neq i, j$. This property underlies the model's tractability, but it places considerable restrictions on substitution patterns that are often at odds with empirical evidence – recall the "red bus-blue bus" paradox due to Debreu (1960).

Two common demand systems are special cases of the structure described above. First, the multinomial logit (MNL) demand system is obtained by setting $h_i(p_i) = \exp\left(\frac{a_i - p_i}{\lambda}\right)$ with $a_i \in \mathbb{R}$ and $\lambda > 0$. Second, the constant elasticity of substitution (CES) demand system is obtained by setting $h_i(p_i) = a_i p_i^{1-\sigma}$ with $a_i > 0$ and $\sigma > 1$. By contrast, nested logit and nested CES demand systems are *not* special cases; Nocke and Schutz (2025) show how the analysis below can be extended to nested demand systems with richer substitution patterns across goods.

Firms and Production. Good $i \in \mathcal{N}$ can be produced at constant marginal cost $c_i > 0$.[3] Given prices p, the profits earned by firm f are

$$\Pi^f(p) = \sum_{i \in f} (p_i - c_i) \hat{D}_i(p_i, H(p)) \equiv \hat{\Pi}^f(p^f, H(p)).$$

Here $p^f \equiv (p_i)_{i \in f}$ denotes the vector of firm f's prices.[4] Firms compete in prices, so that firm f holds all remaining prices p^{-f} fixed when solving

$$\max_{p^f} \ \Pi^f(p). \tag{2}$$

Equilibrium. An *equilibrium* is a price vector p^* such that p^{*f} solves the profit maximization problem (2) with $p^{-f} = p^{*-f}$ for each firm $f \in \mathcal{F}$.

2 Characterizing Equilibrium

In an interior equilibrium, each price p_i must satisfy the first-order condition corresponding to the firm's problem (2):

$$\begin{aligned} 0 = \frac{\partial \Pi^f(p)}{\partial p_i} &= \hat{D}_i + (p_i - c_i) \frac{\partial \hat{D}_i}{\partial p_i} + \left(\sum_{j \in f} (p_j - c_j) \frac{\partial \hat{D}_j}{\partial H} \right) \frac{\partial H}{\partial p_i} \\ &= \hat{D}_i \left[1 - \frac{p_i - c_i}{p_i} \left| \frac{\partial \log(\hat{D}_i)}{\partial \log(p_i)} \right| + \left(\sum_{j \in f} (p_j - c_j) \frac{\partial \hat{D}_j}{\partial H} \right) \frac{\partial H / \partial p_i}{\hat{D}_i} \right]. \end{aligned}$$

Rearranging yields

$$\frac{p_i - c_i}{p_i} \left| \frac{\partial \log(\hat{D}_i)}{\partial \log(p_i)} \right| = 1 + \sum_{j \in f} (p_j - c_j) \frac{\partial \hat{D}_j}{\partial H} \frac{\partial H / \partial p_i}{\hat{D}_i}. \tag{3}$$

This equation is critical for the characterization of equilibrium, and it will be convenient to make two substitutions. First, denote the "monopolistic" elasticity of demand (holding the

[3] (Dis-)Economies of scale or scope can be incorporated with some additional complications.

[4] Throughout, I ignore the possibility that the firm might find it optimal to set an infinite price, equivalent to removing a product from the market.

level of the aggregator H fixed) for good i by

$$\iota_i(p_i) \equiv \left|\frac{\partial \log(\hat{D}_i)}{\partial \log(p_i)}\right| = -p_i \frac{h_i''(p_i)}{h_i'(p_i)}.$$

Second, note that we can explicitly calculate

$$\frac{\partial \hat{D}_j}{\partial H}\frac{\partial H/\partial p_i}{\hat{D}_i} = \frac{h_i'(p_i)}{H(p)^2}\left(-\frac{H(p)}{h_i'(p)}\right)h_i'(p) = \hat{D}_i.$$

Then the first-order condition (3) can then be written

$$\frac{p_i - c_i}{p_i}\iota_i(p_i) = 1 + \hat{\Pi}^f(p^f, H(p)).$$

The key observation here is that the right side of this equation is independent of the good i: Optimal pricing requires that the firm choose a Lerner index $\frac{p_i-c_i}{p_i}$ for each good i equal to a *uniform* "markup" $\mu^f > 1$ above the Lerner index that it would choose under monopolistic competition:

$$\frac{p_i - c_i}{p_i}\iota_i(p_i) = \mu^f. \tag{4}$$

The factor μ^f is called the firm's ι-*markup*. Intuitively, an oligopolistic firm recognizes the competition-softening impact of its prices on the market-level aggregator $H(p)$, so it chooses higher prices than under monopolistic competition.

The ι-markup is a useful object for analysis because it reduces the $|f|$-dimensional pricing problem for firm f to the one-dimensional choice of an ι-markup μ^f. In particular, provided that the left side of (4) is invertible in p_i, this condition can be used to determine the price of each good $i \in f$ as a function of μ^f. Suppose this holds, and denote the inverse of $p_i \mapsto \frac{p_i-c_i}{p_i}\iota_i(p_i)$ by the *pricing function* $r_i(\cdot)$. Given an ι-markup μ^f, the firm's optimal price vector is then given by $p^f = (r_i(\mu^f))_{i\in f}$.

To determine the optimal ι-markup given an arbitrary value for the aggregator H, we can simply substitute μ^f into the first-order condition (4):

$$\mu^f = 1 + \hat{\Pi}^f((r_i(\mu^f))_{i\in f}, H). \tag{5}$$

Suppose that this equation has a unique fixed point, and denote it by the *fitting-in function* $m^f(H)$. Given the value of the aggregator H, $m^f(H)$ is firm f's optimal ι-markup, which pins down its optimal prices p^f. To solve for the equilibrium value of the aggregator, we can simply recall its definition (1):

$$H = \sum_{f \in \mathcal{F}} \sum_{i \in f} h_i(r_i(m^f(H))) + H^0 \equiv \Gamma(H). \tag{6}$$

Here $\Gamma(H)$ denotes the *aggregate fitting-in function*.

To recap, the following procedure characterizes all of the model's equilibria:

1. Find all fixed points H^* of the aggregate fitting-in function, $H^* = \Gamma(H^*)$. Each fixed point corresponds to an equilibrium.

2. Given H^*, the optimal ι-markup for firm f is $\mu^{*f} = m^f(H^*)$.

3. Given μ^{*f}, the optimal prices for firm f are $p^{*f} = (r_i(\mu^{*f}))_{i \in f}$.

This procedure is valid provided that the pricing functions $r_i(\cdot)$ and the fitting-in functions $m^f(\cdot)$ are well-defined. The following assumption, commonly referred to as "Marshall's Second Law of Demand", guarantees that this holds:

Assumption 1. For each good $i \in \mathcal{N}$, the monopolistic demand elasticity $\iota_i(p_i)$ is non-decreasing in p_i when $\iota_i(p_i) > 1$.

Intuitively, this assumption guarantees that the left side of (4) is strictly increasing in p_i and hence invertible on the relevant domain for p_i. Violations of this assumption can easily lead to non-existence of a solution even to a monopoly pricing problem. The following theorem shows that Assumption 1 alone guarantees that the pricing and fitting-in functions are well-defined and that an equilibrium exists.[5]

Theorem 1. *Given Assumption 1, an equilibrium exists for every $\mathcal{F}$ and every $(c_i)_{i \in \mathcal{N}}$.*

In equilibrium, consumer surplus is $\log(H^*)$, firm f's profit is $m^f(H^*) - 1$, and the price of good $i \in f$ is $r_i(m^f(H^*))$.

To facilitate the discussion of economic implications in the next section, I conclude by informally stating two results about the pricing and fitting-in functions.

Proposition 1. *The pricing function $r_i(\cdot)$ is strictly increasing.*

[5] Assumption 1 also implies that the first-order conditions (3) are sufficient to characterize optimality in the firm's problem (2).

Proposition 2. *The fitting-in function $m^f(\cdot)$ is strictly decreasing.*

Note that these propositions imply that firms price more aggressively (i.e., set lower ι-markups and hence lower prices) as competition intensifies (i.e., as the aggregator H increases).

3 Economic Implications

3.1 Comparing Equilibria

Propositions 1 and 2 immediately imply the following result on the ranking of equilibria according to consumer surplus and firm profits:

Proposition 3. *Consider two equilibria with aggregators $H_1^* < H_2^*$. Consumer surplus is strictly higher in the second equilibrium. Each firm makes strictly greater profits in the first equilibrium.*

3.2 Welfare

Like all models of imperfect competition, this model features two potential sources of inefficiency: First, firms price above marginal cost, leading to inefficient under-provision of all goods in equilibrium. Second, conditional on the level of consumer surplus $\log(H^*)$, some firms produce too much while other firms produce too little. To see this, consider the problem of a (constrained) social planner who seeks to maximize aggregate welfare over prices p subject to the constraint that consumer surplus equal an exogenous value $\log(H^*)$:

$$\max_p \sum_f \Pi^f(p) + \log(H^*) \quad \text{subject to} \quad \log(H(p)) = \log(H^*).$$

Letting Λ^* denote the Lagrange multiplier on the constraint, the corresponding first-order condition for p_i is

$$\frac{p_i - c_i}{p_i}\iota(p_i) \;=\; 1 - \Lambda^* + \sum_{f \in \mathcal{F}} \hat{\Pi}^f(p^f, H^*)$$

Comparing to the equilibrium condition (4), we can observe that the social planner would set a *uniform* ι-markup μ^* across firms, satisfying the fixed-point condition

$$\mu^* \;=\; 1 - \Lambda^* + \sum_{f \in \mathcal{F}} \hat{\Pi}^f((r_i(\mu^*))_i, H^*).$$

Equality of ι-markups is a knife-edge condition in equilibrium, indicating inefficiency even conditional on the level of consumer surplus $\log(H^*)$.

3.3 Comparative Statics

Comparative statics around an equilibrium can be performed in a three-step procedure, mirroring the characterization of equilibrium:

1. For a given parameter shift, determine the effects on the pricing functions $r_i(\cdot)$ and the fitting-in functions $m^f(\cdot)$ directly from their definitions.

2. Given the parameter shift and the resulting effects on the pricing and fitting-in functions, use the fixed-point equation (6) to determine the change in the equilibrium value of the aggregator H^*.

3. Use the characterization of equilibrium prices in terms of the (shifted) pricing and fitting-in functions to determine the equilibrium change in all other endogenous variables.

This procedure naturally hinges on the aggregative nature of the pricing game: All equilibrium objects can be readily computed given the equilibrium value of the aggregator H^*.

To see an example, consider an increase in the value of the outside option H^0. This parameter change has no effect on the pricing or fitting-in functions, but it directly raises the aggregate fitting-in function $\Gamma(H)$ defined in (6). Starting from any *stable* solution H^* to the aggregate fixed-point equation (6) (i.e., a solution with $\Gamma'(H^*) < 1$), the equilibrium value of the aggregator H^* must increase. As a result, Propositions 1 and 2 immediately imply that all prices and firm profits fall while consumer surplus increases. The equilibrium responds similarly to the introduction of a new firm. Comparative statics with respect to marginal costs c_i are more nuanced – see Section 3.4 of Nocke and Schutz (2018).

4 Special Cases: MNL and CES Demand

4.1 Type Aggregation

As noted above, the MNL and CES demand systems are special cases of the general demand system described in Section 1: MNL demand is recovered by setting $h_i(p_i) = \exp\left(\frac{a_i - p_i}{\lambda}\right)$, while CES demand obtains with $h_i(p_i) = a_i p_i^{1-\sigma}$. Beyond their ubiquity in theoretical and applied research in economics, these demand systems are of interest because they exhibit a property called *type aggregation* that makes equilibrium analysis even more tractable.

To define this property, first define firm f's *type* T^f under each demand system as follows:

$$\begin{aligned}
(\text{MNL}) \quad T^f &\equiv \sum_{i \in f} \exp\left(\frac{a_i - c_i}{\lambda}\right), \\
(\text{CES}) \quad T^f &\equiv \sum_{i \in f} a_i c_i^{1-\sigma}.
\end{aligned}$$

In each case, the firm's type is simply its contribution to the aggregator H under marginal cost pricing. The type is naturally increasing in the number of products owned by the firm, the firm's productivities $1/c_i$, and the firm's qualities a_i.

Now under each demand system, the fixed-point equation (5) for the ι-markup μ^f can be simplified to

$$\begin{aligned}
(\text{MNL}) \quad \mu^f \left(1 - \frac{T_f}{H} \exp\left(-\mu^f\right)\right) &= 1, \\
(\text{CES}) \quad \mu^f \left(1 - \frac{\sigma - 1}{\sigma} \frac{T^f}{H} \left(1 - \frac{\mu^f}{\sigma}\right)^{\sigma - 1}\right) &= 1.
\end{aligned}$$

Crucially, in each case the equation defining the firm's ι-markup depends on the firm f and the aggregator H only through the value T_f/H. We can then define a function $m(\cdot)$ such that $\mu^f = m(T^f/H)$ is firm f's optimal ι-markup given its type T^f and the value of the aggregator H; in essence, the firm's fitting-in function is simply $m^f(H) = m(T^f/H)$. *Type aggregation* is the observation that fitting-in functions depend on firm identity only through the type T^f.

Type aggregation allows a more intuitive characterization of equilibrium, similar to that in the Cournot model. To see this, define the *share* of a good $i \in \mathcal{N}$ by

$$s_i \equiv \frac{h_i(p_i)}{H}.$$

The value s_i corresponds to good i's share by volume under MNL demand, while it corresponds to good i's share by revenue under CES demand. Firm f's share is simply the sum of its goods' shares, $s^f \equiv \sum_{i \in f} s_i$. Setting prices p^f according to the ι-markup condition (4), it can be shown that firm shares satisfy

$$\text{(MNL)} \quad s^f = S\left(\frac{T^f}{H}\right) \equiv \frac{T^f}{H}\exp\left(-m\left(\frac{T^f}{H}\right)\right),$$

$$\text{(CES)} \quad s^f = S\left(\frac{T^f}{H}\right) \equiv \frac{T^f}{H}\left(1-\frac{m\left(\frac{T^f}{H}\right)}{\sigma}\right)^{\sigma-1}.$$

In each case, a firm's share is determined entirely by the scalar T^f/H. Our observations thus far demonstrate that with MNL or CES demand, a firm's ι-markup (i.e., its pricing behavior), profits, and market share are determined entirely by its type T^f, holding the value of the aggregator H fixed. The equilibrium value of the aggregator H^* can be determined by the requirement that firm shares sum to $1-\frac{H^0}{H}$:

$$\sum_{f\in\mathcal{F}} S\left(\frac{T^f}{H^*}\right) = 1-\frac{H^0}{H^*}.$$

This is exactly the aggregate fixed-point equation (6). The following proposition summarizes key properties and comparative statics:

Proposition 6. *With MNL or CES demand,*

1. *firm f's ι-markup $m(T^f/H)$, profits $m(T^f/H)-1$, and market share $S(T^f/H)$ are all increasing in T^f/H;*

2. *equilibrium consumer surplus* $\log(H^*)$ *and aggregate welfare are both increasing in T^f for each $f \in \mathcal{F}$; and*

3. *for each firm $f \in \mathcal{F}$, every other firm g's equilibrium ι-markup, profits, and market share are decreasing in T^f.*

Note the result that aggregate welfare is increasing in T^f is not obvious: Aggregate welfare can decline in response to a decrease in a firm's marginal cost on the Cournot model.

4.2 Application: Horizontal Merger Analysis

To see a (brief!) application of this framework to merger analysis, suppose a MNL demand system, and consider a horizontal merger between firms f and g to form a new firm M.[6] The

[6] See Nocke and Whinston (2022) and Nocke and Schutz (2025) for much more detailed analyses.

merged firm M owns all products $i \in f \cup g \equiv M$, and it may achieve *merger efficiencies* that lead to higher qualities a_i or lower marginal costs c_i for $i \in M$.

Are merger efficiencies necessary to prevent consumer harm from the merger? Before the merger, the equilibrium value of the aggregator H^* satisfies

$$S\left(\frac{T^f}{H^*}\right)+S\left(\frac{T^g}{H^*}\right)+\sum_{f'\neq f,g} S\left(\frac{T^{f'}}{H^*}\right) = 1-\frac{H^0}{H^*}.$$

In order to maintain consumer surplus of at least $\log(H^*)$, the merged firm's type M must be weakly greater than $\bar{T}^M$, where

$$S\left(\frac{\bar{T}^M}{H^*}\right)+\sum_{f'\neq f,g} S\left(\frac{T^{f'}}{H^*}\right) = 1-\frac{H^0}{H^*}.$$

Subtracting the previous two equations, the CS-neutral type $\bar{T}^M$ satisfies

$$S\left(\frac{\bar{T}^M}{H^*}\right) = S\left(\frac{T^f}{H^*}\right)+S\left(\frac{T^g}{H^*}\right).$$

Merger synergies are required for this equation to hold if and only if $\bar{T}^M > T^f + T^g$. This follows provided that the share function $S(\cdot)$ is *strictly sub-additive*:

$$S\left(\frac{T^f + T^g}{H^*}\right) < S\left(\frac{T^f}{H^*}\right)+S\left(\frac{T^g}{H^*}\right).$$

But this follows immediately from the closed-form expression for the share function $S(\cdot)$ with MNL demand, which is strictly increasing, strictly concave, and satisfies $S(0) = 0$. These conditions imply that $S(\cdot)$ is strictly sub-additive, so that merger efficiencies are necessary to avoid consumer harm.

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