14.452 Recitation 2

Uzawa, Solow, Growth Regressions

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These slides build on work by past 14.452 TAs: Shinnosuke Kikuchi, Joel Flynn, Karthik Sastry, Ernest Liu, Ludwig Straub, . . .

Admin

- Problem Set 2 already posted, due **November 8 at 2:30pm**
  - solving the Solow model! comparative statics! Uzawa!
  - only problems 1 and 3 must be turned in (we'll start 4 today)
- Problem Set 3 to be posted Nov 8, not due until Nov 22

Plan for today

1. Uzawa's Theorem redux
2. **[Lecture]** Levels regressions: how well does Solow explain income differences?
3. Dynamics in the continuous-time Solow model
4. Growth regressions: do richer or poorer countries grow more quickly?

Uzawa's Theorem

The Solow/neoclassical setup

- Time $t \in [0, \infty)$ (but let's keep t implicit to simplify notation)
- Production $Y = F(K, L, A)$, CRS in (K, L)
- Market clearing: $Y = C + I$
- Capital accumulation: $\dot{K} = I - \delta K$
- Population growth: $\dot{L} = Ln$
- *No assumptions about (consumer) behavior!*

Modeling technological change

If technological change $\dot{A}$ drives growth, how should we model it?

- Idea: make sure we can match some stylized (Kaldor) facts about long-run growth
- This rules out many kinds of technological change
  - e.g., cannot have $Y = AK^{\alpha}L^{1-\alpha}$ with $\dot{A} = gA^{1+\phi}$ for $\phi > 0 \longrightarrow$ why?
- Uzawa's Theorem(s): how much "bite" do our balanced growth assumptions have for how we can model long-run technological change?

  a ton

Theorem (Uzawa I)

Given our setup, suppose for all $t \geq 0$

$$\frac{\dot{Y}}{Y} = g_Y > 0, \quad \frac{\dot{K}}{K} = g_K > 0, \quad \frac{\dot{C}}{C} = g_C > 0.$$

Then $g_Y = g_K = g_C$, and there exists a CRS production function $\hat{F}$ such that

$$Y(t) = \hat{F}\left(K(t), \hat{A}(t)L(t)\right) \quad \text{for all } t \geq 0,$$

where $\dot{\hat{A}} = g\hat{A}$ and $g = g_Y - n$.

- Only empirical "facts" used here are constant growth rates (*not constant shares*)
- Assuming constant growth after $t = 0$, but can also state this just for constant growth after some finite t
- Can also prove a theorem about "asymptotic" constant growth: $\lim_{t\to\infty} \frac{\dot{X}}{X} = g_X$

Proving Uzawa I

Part 1: Equal growth rates

This is implied just by market clearing (MC) and capital accumulation (CA)

$$
\begin{aligned}
\text{CA} \quad &\Longrightarrow \quad g_K = \frac{\dot{K}}{K} = \frac{I}{K} - \delta \\
\text{MC} + \text{CA} \quad &\Longrightarrow \quad \frac{Y}{K} = \frac{C}{K} + \frac{I}{K} \\
&\qquad\qquad = \frac{C}{K} + g_K + \delta
\end{aligned}
$$

Let's re-write the last equation, recalling $X(t) = X(0)\exp(g_X t)$

$$
\frac{Y(0)}{K(0)}\exp\left((g_Y - g_K)t\right) = \frac{C(0)}{K(0)}\exp\left((g_C - g_K)t\right) + g_K + \delta
$$

Proving Uzawa I

$$\frac{Y(0)}{K(0)}\exp\left((g_Y - g_K)t\right) = \frac{C(0)}{K(0)}\exp\left((g_C - g_K)t\right) + g_K + \delta$$

When can this equation hold for all $t \geq 0$?

1. $g_Y = g_K = g_C$ (*nice!*)
2. $g_Y = g_K \neq g_C$ and $C(0) = 0 \longrightarrow$ contradicts $g_C > 0$
3. $g_Y \neq g_K = g_C$ and $Y(0) = 0 \longrightarrow$ contradicts $g_Y > 0$
4. $g_Y = g_C$ and $Y(0) = C(0) \longrightarrow$ contradicts $g_K > 0$

Note: this is really all about what it means for sums of exponentials to be equal (need equal growth rates)

Proving Uzawa I

Part 2: Labor-augmenting representation

This is implied by $g_Y = g_K$, the CRS aggregate production function, and constant population growth

Start with $Y(0) = F(K(0), L(0), A(0))$, and multiply both sides by $\exp(g_Y t)$:

$$Y(t) = F\left(K(0)\exp(g_Y t), L(0)\exp(g_Y t), A(0)\right)$$

Using $K(t) = K(0)\exp(g_Y t)$ and $L(t) = L(0)\exp(nt)$,

$$Y(t) = F\left(K(t), L(t)\exp((g_Y - n)t), A(0)\right)$$

Proving Uzawa I

$$Y(t) = F\left(K(t), L(t)\exp\left((g_Y - n)t\right), A(0)\right)$$

This is the representation we want!

- $\hat{F}\left(K, \hat{A}L\right) = F\left(K, \hat{A}L, A(0)\right)$, where $\hat{A}(t) = \exp\left((g_Y - n)t\right)$
- Note that we never evaluated $A(t)$ at $t > 0$
    - the original production function F might produce constant growth in *crazy ways*
    - who knows what's going on with factor shares, interest rates, wages, etc.

Uzawa II

Caveat to Uzawa I: no reason for factor prices/shares to behave similarly under $\hat{F}$ and F

Theorem (Uzawa II)

Under the same assumptions as Uzawa I, if factor markets are competitive, then $R(t) = R^*$ if and only if F and $\hat{F}$ have the same marginal products at all $t \geq 0$.

- The economy with labor-augmenting technology is *observationally equivalent* to the original economy on the balanced growth path

$\Rightarrow$ can just work with labor-augmenting technology if we're just interested in BGPs

Questions before we look at an example?

Example: PS2 Question 4.1

- Take the standard Solow model with no usual technological change, $A(t) = A$
- But modify the capital accumulation equation to $\dot{K}(t) = q(t) I(t) - \delta K(t)$, where $q(t)$ varies exogenously ($\approx$ inverse of relative price of machines to output)
- Suppose $\dot{q}/q = \gamma_K > 0$
- For what production functions $F(K, L)$ does there exist a "steady state equilibrium"?

Note: not *exactly* Uzawa because we'll use the constant savings rate s, but same idea

Getting started

- Can prove this just using $I(t) = sY(t)$ and the capital accumulation equation
- With $k = K/L$, these equations imply $\dot{k} = qsf(k) - (n + \delta)k$
- Suppose a "steady state" with $\dot{k}/k = g_k \geq 0$.Then

$$k(0)g_k \exp(g_k t) = sq(0)\exp(\gamma_K t) f(k(0)\exp(g_k t)) - (n + \delta) k(0)\exp(g_k t)$$

- Simplifying a bit:

$$\frac{k(0)}{sq(0)}(g_k + n + \delta)\exp((g_k - \gamma_K)t) = f(k(0)\exp(g_k t))$$

Concluding

$$\frac{k(0)}{sq(0)}\left(g_k+n+\delta\right)\exp\left(\left(g_k-\gamma_K\right)t\right)=f\left(k(0)\exp\left(g_k t\right)\right)$$

- Must have $g_k > 0$ for this equation to hold
- But then this equation *pins down* f *at any* $k > k(0)$
- For arbitrary k, let $t = \frac{1}{g_k}\log\left(\frac{k}{k(0)}\right)$. Then

$$f\left(k\right)=\frac{k(0)^{\frac{\gamma_K}{g_k}}}{sq(0)}\left(g_k+n+\delta\right)k^{\frac{g_k-\gamma_K}{g_k}}$$

- This only holds if F is Cobb-Douglas with capital share $\frac{g_k-\gamma_K}{g_k}$!
- *Why did this happen?* Same capital accumulation dynamics as an economy with production technology $q(t)F(K,L) \longrightarrow$ Uzawa's revenge!

Solow Model

Solow model redux

- Solow model = Uzawa I setup + constant savings rate s:

$$I(t) = sY(t)$$

- Let's also assume labor-augmenting technology $Y = F(K, AL)$ with $\dot{A}/A = g$

- Model has one state variable $k = K/AL$ with *backward-looking dynamics*:

$$\dot{k} = sf(k) - (\delta + n + g)\,k$$

- BGP is just a steady-state for k:

$$\dot{k} = 0 \quad \Longleftrightarrow \quad sf(k^*) = (\delta + n + g)\,k^*$$

Off-BGP dynamics

What happens away from the BGP?

$$k(t) < k^*: \quad \text{concavity of } f \quad \Rightarrow \quad \dot{k} > 0, \qquad \textbf{capital deepening}$$
$$\Rightarrow \frac{d\log(y)}{dt} > g$$

$$k(t) > k^*: \quad \text{concavity of } f \quad \Rightarrow \quad \dot{k} < 0, \qquad \textbf{capital ``shallowing''}$$
$$\Rightarrow \frac{d\log(y)}{dt} < g$$

- With $k(t) \neq k^*$, Solow model only features balanced growth *asymptotically* $(t \to \infty)$
- Important prediction of the model: given two countries with the same fundamentals $\{f, s, \delta, n, g\}$, the country with the smaller k grows faster

Speed of convergence: example

- Solow model has *quantitative* implications for speed of convergence to BGP
- Before doing this generally, let's take a look at an example:

$$F(K, AL) = K^{\alpha}(AL)^{1-\alpha} \quad \Rightarrow \quad f(k) = k^{\alpha}$$

- Steady-state equation: $s\,(k^*)^{\alpha} = (\delta + n + g)\,k^*$
- Can solve directly for all quantities in the BGP:

$$k^* = \left(\frac{s}{\delta + n + g}\right)^{\frac{1}{1-\alpha}}, \quad \hat{y}^* = \left(\frac{s}{\delta + n + g}\right)^{\frac{\alpha}{1-\alpha}}, \quad c^* = (1 - s)\left(\frac{s}{\delta + n + g}\right)^{\frac{\alpha}{1-\alpha}}$$

Speed of convergence: example

- Even better, we can solve for the entire **path** of k away from the BGP
- Idea: accumulation equation $\dot{k} = sk^{\alpha} - (\delta + n + g)\, k$ is *almost* linear in k

  Try the change of variables $x = k^{1-\alpha}$, and see if we can get a nice equation for $\dot{x}$:

$$
\begin{aligned}
\dot{x} &= (1-\alpha)\, k^{-\alpha} \dot{k} \\
&= (1-\alpha)\, k^{-\alpha} \left[sk^{\alpha} - (\delta + n + g)\, k \right] \\
&= (1-\alpha)\, s - (1-\alpha)(\delta + n + g)\, x \qquad (\textit{linear!})
\end{aligned}
$$

Speed of convergence: example

- Cookie-cutter formula for integrating a linear ODE gives

$$x(t) = \frac{s}{\delta + n + g} + \left[x(0) - \frac{s}{\delta + n + g}\right] \exp\left(-\left(1-\alpha\right)\left(\delta + n + g\right) t\right)$$

- Substituting $x(t) = k(t)^{1-\alpha}$ and k^*, we can rearrange to find

$$\frac{k(t)^{1-\alpha} - (k^*)^{1-\alpha}}{k(0)^{1-\alpha} - (k^*)^{1-\alpha}} = \exp\left(-\left(1-\alpha\right)\left(\delta + n + g\right) t\right)$$

- *So what?*
  - the gap between current k and BGP k^* closes at an *exponential rate*
  - the convergence rate is *decreasing* in α: less severe diminishing returns to K at each t

Speed of convergence

- What if F isn't Cobb-Douglas? Does some version of these conclusions hold?

  Yes, at least close to the BGP

- Easiest to see this by linearizing $\dot{k}$ around k^*:

$$\dot{k} = sf(k) - (\delta + n + g)\,k$$

$$\approx \left.\dot{k}\right|_{k=k^*} + sf'(k^*)(k - k^*) - (\delta + n + g)(k - k^*)$$

$$= 0 + sf'(k^*)(k - k^*) - (\delta + n + g)(k - k^*)$$

- Convergence is again **exponential**: $\frac{d}{dt}|k - k^*|$ is increasing in $k - k^*$

Rewrite: log changes

- Exponential convergence $\Rightarrow$ more convenient to write equation in logs

$$\frac{1}{k}\frac{dk}{dt} = \frac{d\log(k)}{dt}$$

- Algebra + steady-state condition + approximation $\log\left(\frac{k^*}{k}\right) \approx \frac{k^*}{k} - 1$ gives

$$\frac{d\log(k)}{dt} \approx -\left(1-\varepsilon(k^*)\right)(\delta+n+g)\left(\log(k)-\log(k^*)\right),$$

where $\varepsilon(k) = \frac{d\log(f(k))}{d\log(k)} = \frac{k}{f(k)}f'(k)$

- Compare this to the (exact) equation in the Cobb-Douglas case!

$$\frac{dk^{1-\alpha}}{dt} = -(1-\alpha)(\delta+n+g)\left(k^{1-\alpha}-(k^*)^{1-\alpha}\right)$$

- Higher $\varepsilon \Rightarrow$ slower convergence, and divergence at $\varepsilon = 1$! (AK model)

Output convergence

One last piece of algrebra (*I promise!*)...

- Can equivalently express convergence in $y = Y/L$ instead of k:

$$\frac{d\log(y)}{dt} \approx g - \underbrace{(1-\varepsilon(k^*))(\delta+n+g)}_{\text{“convergence coefficient”},\ b^1}(\log(y) - \log(y^*(t))),$$

  where $y^*(t) = A(t)f(k^*)$

- How to interpret the convergence coefficient b^1?

  Suppose $\varepsilon = 0.33$, $\delta = 5\%$, $n = 1\%$, $g = 2\%$

  $\Rightarrow b^1 = 0.0536$, “income gap closes at 5% per year”

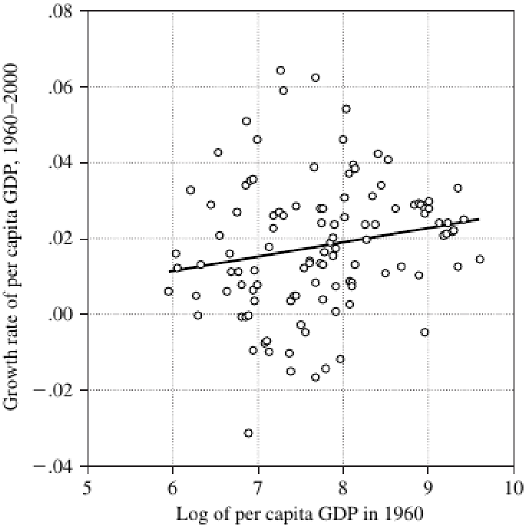
Growth regressions

- Output convergence equation motivates a style of empirics following Barro (1991):

$$\Delta \log(y)_{ct} = b_c^0 + b_c^1 \log(y_{ct-1}) + u_{ct}$$

- Existence of (and convergence to) unique steady state requires $b_c^1 < 0$
- First suppose $b_c^0 = b^0$ and $b_c^1 = b^1$. Justifications?
  1. we think all countries in our dataset have the same fundamentals $\{f, s, \delta, n, g\}$
  2. we want an easy correlation interpretation: $\hat{b}^1 < 0 \iff$ poorer countries have faster growth than rich countries on average
  3. does $b_c^1 < 0$ necessarily imply $\hat{b}^1 < 0$?
- Do we find $\hat{b}^1 < 0$ in the data? Does this matter if the model doesn't fit well?

Barro & Sala-i-Martin (2004): unconditional divergence



Conditional convergence

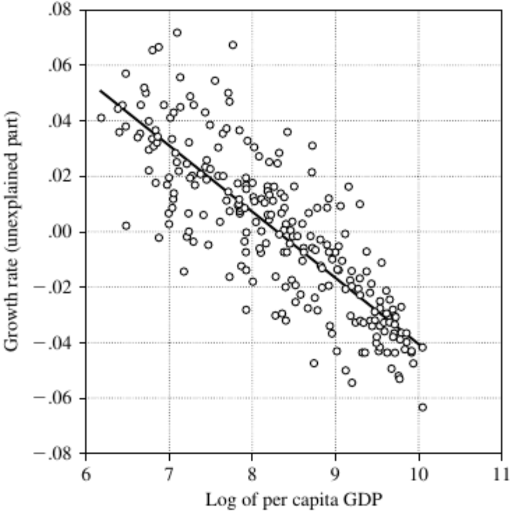
- Lack of unconditional convergence motivated the idea of *conditional convergence*:

$$\Delta \log(y)_{ct} = X_{ct}^T \beta + b^1 \log(y_{ct-1}) + u_{ct}$$

- Generally find convergence conditional on "correlates" X_{ct} (investment rate, education, institutions, fertility, . . .)

- But **many econometric issues** and **very difficult to interpret**

Barro & Sala-i-Martin (2004): conditional convergence



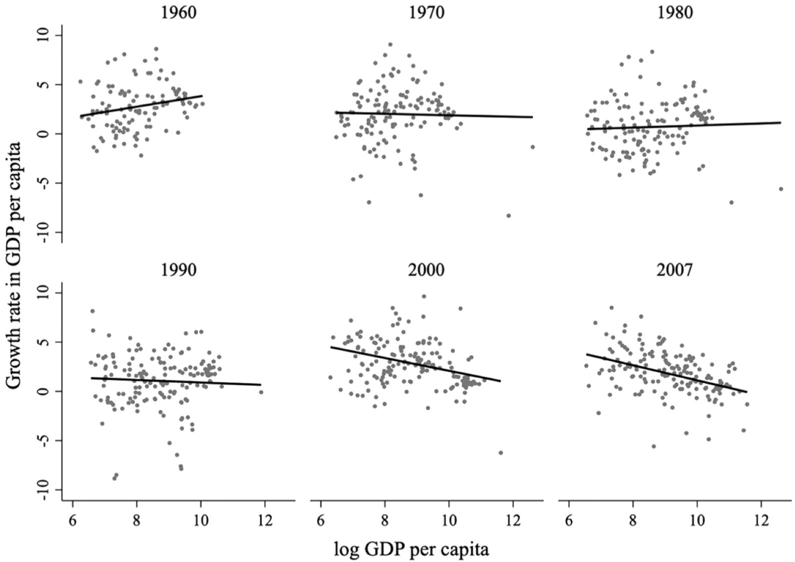
Recent changes?

- At least two recent papers (Patel, Sandefur, & Subramanian 2021; Kremer, Willis, & You 2022) suggest that we might be trending toward *unconditional* convergence
  - but see the criticism of Acemoglu & Molina (2022)
- Kremer, Willis & You (2022) estimate

$$\Delta \log(y)_{ct} = b_t^0 + b_t^1 \log(y_{ct-1}) + u_{ct}$$

  - Δ is over 10-year intervals
  - b_t^0, b_t^1 vary with the beginning of the interval
  - How does b_t^1 change over time?

Unconditional convergence?



Questions?