14.452 Recitation 3

NGM, Stochastic Growth, q-Theory

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These slides build on work by past 14.452 TAs: Shinnosuke Kikuchi, Joel Flynn, Karthik Sastry, Ernest Liu, Ludwig Straub, . . .

Plan for today

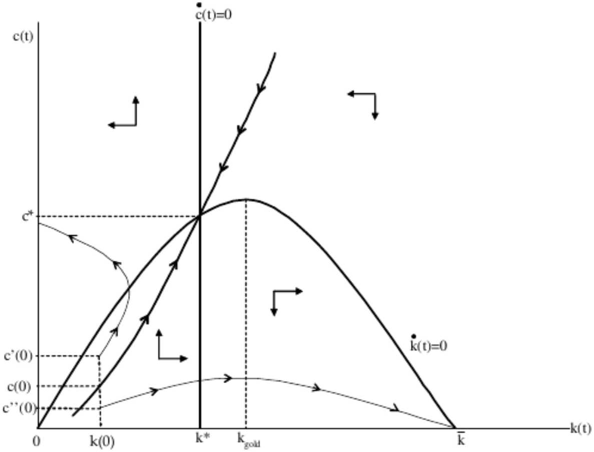
1. **[Lecture]** Calibration excercises with the NGM
2. Discrete-time NGM
   - practice problem: comparative statics/dynamics
3. Stochastic growth (Brock-Mirman)
4. Continuous-time optimal control practice (q-theory)

A motivating question

Brock and Mirman (*JET*, 1972)
"Optimal Economic Growth and Uncertainty: The Discounted Case"

The cornerstone of one-sector optimal economic growth models is the existence and stability of a steady-state solution for optimal consumption policies. The optimal consumption policy is the stable branch of the saddle point solution of the system of differential equations governing the dynamics of the economy. Examples of this type of behavior can be found in Cass [2] and Koopmans [4]. However, the stable branch solution is a knife-edge policy in the sense that any perturbation, no matter how small, results in instability and eventual annihilation. (This phenomenon is true when the Euler conditions are adhered to after the perturbation). Small pertubations might occur due to observation errors or the lack of knowledge of the exact production functions. It seems reasonable to expect that all sorts of human errors influence decision variables. Hence unless perfect knowledge of all variables, present and future, were known with complete certainty, and unless all decisions were made with exactness, economies of the one-sector deterministic type would lead at best to suboptimal consumption and investment policies.

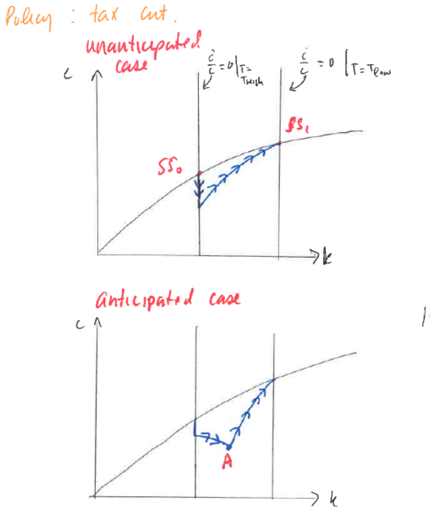
So far: dynamical system characterization



Good for seeing:

- Saddle-path behavior
- Transitional dynamics

Comparative dynamics are mysterious



Even writing down this problem is gross, which makes it hard to contemplate deviations

What if we miss the future tax rate's path?

Dynamic programming approach will make this more clear

Discrete-Time Neoclassical Growth

Basic structure

- Fixed labor $l_t = 1$
- Investment technology:

$$k_{t+1} = (1 - \delta) k_t + f(k_t) - c_t$$

- Assume $f(\cdot)$ is differentiable, strictly concave, strictly increasing, satisfies Inada
- Representative household with $\sum_{t=0}^{\infty} \beta^t u(c_t)$

A planner's problem

- Problem can be written as

$$\max_{\{k_{t+1},c_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u\left(c_t\right)$$

$$\text{s.t. : } \; k_{t+1} = f\left(k_t\right) + \left(1-\delta\right) k_t - c_t, \qquad k_0 > 0 \text{ given and } k_t \geq 0$$

- Want to write this as stationary decision problem (why?)
- **Trick 1**: replace c_t in the objective and reduce the choice variables to k_{t+1} only
  - One state, one choice
- **Trick 2**: argue $k_{t+1} \in G\left(k_t\right)$ is compact-valued
  - i.e., lies in some bounded set

Bounded support for k

- Lower bound is 0 since $k_t \geq 0$
- Upper bound, take:

$$\overline{k} : \ \delta \overline{k} = f\left(\overline{k}\right).$$

  Why?
- Suppose $k_t < \overline{k}$ then:

$$\overline{k} = f\left(\overline{k}\right) + (1 - \delta)\,\overline{k} > f\left(k_t\right) + (1 - \delta)\,k_t = k_{t+1}$$

  so even consuming 0 we don't surpass $\overline{k}$ if $k_t < \overline{k}$

Bounded support for k

- Suppose now $k_t > \overline{k}$ then:

$$k_{t+1} = f(k_t) + (1-\delta) k_t = k_t + f(k_t) - \delta k_t < k_t$$

because $f(k_t) - \delta k_t < 0$ for $k_t > \overline{k}$

- So, even consuming 0, we don't surpass k_t if $k_t > \overline{k}$

- Thus we can take:

$$\begin{aligned} k_{t+1} &\in \left[0, \vec{k}\right] \\ \vec{k} &:= f\left(\max\left\{k_t, \overline{k}\right\}\right) + (1-\delta)\max\left\{k_t, \overline{k}\right\} \end{aligned}$$

New, easier problem

- Thus, we can write the problem as:

$$V(k) = \max_{k_{next} \in [0, \vec{k}]} u\left(f(k) + (1-\delta)k - k_{next}\right) + \beta V(k_{next})$$

- From 14.451 we know: $k_{next} = g(k)$ is continuous and strictly increasing and using Euler + Envelope:

$$u'(c_t) = \beta V'(k_{t+1}) = \beta u'(c_{t+1}) \left[f'(k_{t+1}) + (1-\delta)\right]$$

and the TVC:

$$\lim_{t \to \infty} \beta^t \left[f'(k_t) + (1-\delta)\right] u'(c_t) k_t = 0$$

Solving the model

- Steady state: $k^* = g(k^*)$
- From Euler:

$$u'(f(k^*) + (1-\delta)k^* - k^*)$$
$$= \beta u'(f(k^*) + (1-\delta)k^* - k^*)\left[f'(k^*) + (1-\delta)\right]$$

or

$$1 = \beta\left[f'(k_t) + (1-\delta)\right]$$

- Since $f(\cdot)$ is strictly concave, this condition defines k^* uniquely.

Stability

- Is the steady state globally stable? Yes

- Transitional Dynamics: Suppose $k_t < k^*$. Then

$$\begin{aligned} k_t &< k^* \\ g(k_t) &< g(k^*) \\ k_{t+1} &< k^* \end{aligned}$$

Stability

- Do we know capital is going up when $k < k^*$?
- Suppose (contradiction!) $k_{t+1} < k_t$:

$$\begin{aligned} u'\left(f\left(k_t\right) + (1-\delta)\,k_t - k_{t+1}\right) &= \beta V'\left(k_{t+1}\right) \\ &> \beta V'\left(k_t\right) \\ &= \beta\left[f'\left(k_t\right) + (1-\delta)\right] \\ &\quad \times u'\left(f\left(k_t\right) + (1-\delta)\,k_t - k_{t+1}\right) \\ &> u'\left(f\left(k_t\right) + (1-\delta)\,k_t - k_{t+1}\right) \end{aligned}$$

where we used that $V'(\cdot)$ is strictly decreasing and $\beta\left[f'\left(k_t\right) + (1-\delta)\right] > 1$ for $k_t < k^*$

Stability

- Thus, if $k_t < k^*$ we have $k_{t+1} \in (k_t, k^*)$
- Same ideas if $k_t > k^*$
- Thus, suppose $k_0 \in (0, k^*)$, we have an increasing sequence $k_{t+1} = g(k_t)$ which is bounded above by k^*. Hence, it converges. Since there is a unique positive steady state the limit is k^*.

Turning to CE

- Now, focus in competitive equilibrium (CE) and show both coincide, so steady state and transitional dynamics we derived also work for the CE
- Ultimately, not surprising since conditions for 1st and 2nd Welfare Theorems satisfied

HH problem

The rep. household solves:

$$\max_{\{a_{t+1},c_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t)$$

$$\text{s.t.}: \begin{cases} a_{t+1} = (1+r_t)\, a_t + w_t - c_t, \;\; a_0 > 0 \text{ given} \\ \lim_{t\to\infty} a_t \left(\prod_{s=1}^{t-1} \frac{1}{1+r_s}\right) \geq 0. \end{cases}$$

Shortcut to solution

- From 14.451 we know solution is characterized by:

$$u'(c_t) = \beta(1 + r_{t+1})u'(c_{t+1})$$

and TVC:

$$\lim_{t\to\infty} \beta^t u'(c_t)(1 + r_t)a_t = 0$$

Thinking about the TVC

- Note that using the Euler equation we can rewrite TVC as:

$$
\begin{aligned}
\lim_{t\to\infty} \beta^t u'(c_t)(1+r_t)\, a_t &= 0 \\
\lim_{t\to\infty} \beta^t u'(c_{t-1}) \frac{1}{\beta(1+r_t)}(1+r_t)\, a_t &= 0 \\
\lim_{t\to\infty} \beta^{t-1} u'(c_{t-1})\, a_t &= 0 \\
\lim_{t\to\infty} \beta^{t-1} u'(c_{t-2}) \frac{1}{\beta(1+r_{t-1})} a_t &= 0 \\
\lim_{t\to\infty} \beta^{t-3} u'(c_{t-3}) \frac{1}{(1+r_{t-2})} \frac{1}{(1+r_{t-1})} a_t &= 0 \\
&\vdots \\
\lim_{t\to\infty} u'(c_0) \frac{1}{(1+r_1)} \cdots \frac{1}{(1+r_{t-1})} a_t &= 0
\end{aligned}
$$

Thinking about the TVC

- And thus:

$$\lim_{t\to\infty} u'\left(c_0\right)\left(\prod_{s=1}^{t-1}\frac{1}{(1+r_s)}\right)a_t = 0$$

$$u'\left(c_0\right)\lim_{t\to\infty}\left(\prod_{s=1}^{t-1}\frac{1}{(1+r_s)}\right)a_t = 0$$

$$\lim_{t\to\infty}\left(\prod_{s=1}^{t-1}\frac{1}{(1+r_s)}\right)a_t = 0$$

which is a stronger version of the No-Ponzi condition

Plugging in prices

- Now, we know in the CE $r_t = f^{'}(k_t) - \delta$
- So the Euler equation becomes:

$$u'(c_t) = \beta(1 + r_{t+1})\, u'(c_{t+1})$$
$$u'(c_t) = \beta\left(1 + f'(k_{t+1}) - \delta\right) u'(c_{t+1})$$

which is the same Euler equation from the Optimal Growth Problem!

Same TVC?

- Finally we want to check that the TVC of the CE is the same as the one in the Optimal Growth Problem (OGP)
- In the OGP we had:

$$\lim_{t\to\infty} \beta^t \left[f'(k_t) + (1-\delta)\right] u'(c_t)\, k_t = 0.$$

- In the CE we have:

$$\lim_{t\to\infty} \beta^t [1 + r_t]\, u'(c_t)\, a_t = 0$$

$$\lim_{t\to\infty} \beta^t \left[f'(k_t) + (1-\delta)\right] u'(c_t)\, k_t = 0$$

where we used that in the CE $a_t = k_t$ and $r_t = f'(k_t) - \delta$

Precise equivalence

- Since the CE path coincides with the OGP one, we know that starting from any $k_0 > 0$ the CE path converges monotonically to the unique steady state

- Note that the No-Ponzi condition:

$$\lim_{t\to\infty} a_t \left(\prod_{s=1}^{t-1} \frac{1}{1+r_s} \right) \geq 0$$

ensures that the Arrow-Debreu CE coincides with the sequential trading one

why

NGM Practice Problem

Practice problem

- Standard neoclassical growth model in discrete time $t \in \{0, 1, \ldots\}$, $n = 0$, $L = 1$
- Twist: government pays subsidy $\chi > 0$ per unit of capital rented by firm
- Firm's problem:
$$\max_{K,L} F(K, L) - w_t L - (1 - \chi) r_t L$$
- Subsidy financed by *lump-sum* tax on households τ_t.

  Budget constraint:
$$c_t + k_{t+1} = w_t + r_t k_t + (1 - \delta) k_t - \tau_t, \quad \chi r_t k_t = \tau_t$$
- Household preferences: $u(c) = \frac{c^{1-\theta} - 1}{1-\theta}$

Deriving equilibrium conditions

- Let's derive the equilibrium conditions and compare to the "standard" equilibrium conditions
- Given r_{t+1}, Euler equation: $\frac{c_t^{-\theta}}{c_{t+1}^{-\theta}} = \beta\left[1 + r_{t+1} - \delta\right]$

  In equilibrium, $r_{t+1} = f'(k_{t+1})/(1-\chi)$. Substitute:

$$\frac{c_t^{-\theta}}{c_{t+1}^{-\theta}} = \beta\left[1 + \frac{f'(k_{t+1})}{1-\chi} - \delta\right]$$

- Budget constraint, using the zero-profit condition $f(k_t) = w_t + (1-\chi)r_t k_t$:

$$c_t + k_{t+1} = f(k_t) + (1-\delta)k_t$$

- Standard equilibrium conditions:

$$\begin{aligned}\frac{c_t^{-\theta}}{c_{t+1}^{-\theta}} &= \beta\left[1 + f'(k_{t+1}) - \delta\right] \\ c_t + k_{t+1} &= f(k_t) + (1-\delta)k_t\end{aligned}$$

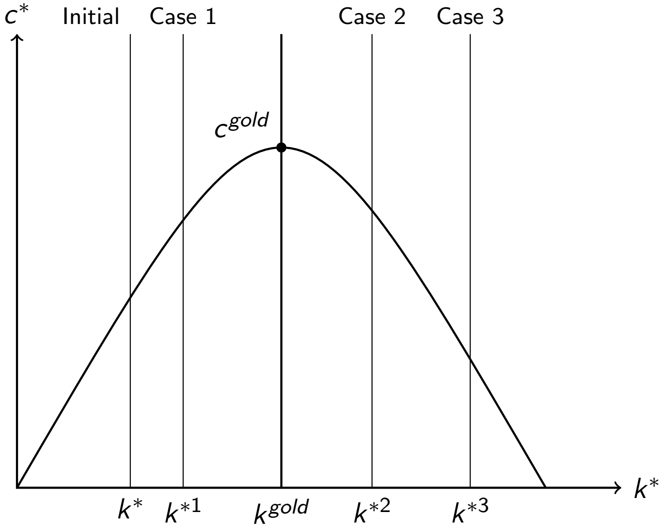
Steady state

- How does χ affect the steady state? Can k^* exceed the golden rule level of the capital stock?

- Steady state conditions:

$$
\begin{aligned}
1 &= \beta\left[1 + \frac{f'(k^*)}{1-\chi} - \delta\right] \\
c^* &= f(k^*) - \delta k^*
\end{aligned}
$$

- Suppose initial steady state $k^* < k^{gold}$, where k^{gold} maximizes $f(k) - \delta k$
  Three cases:
  1. χ is so small that the new steady state $k^{*1} < k^{gold}$, so that c^* increases
  2. χ is not so small that the new steady state $k^{*2} > k^{gold}$, but not so big that c^* increases
  3. χ is so big that the new steady state $k^{*3} > k^{gold}$ and c^* decreases



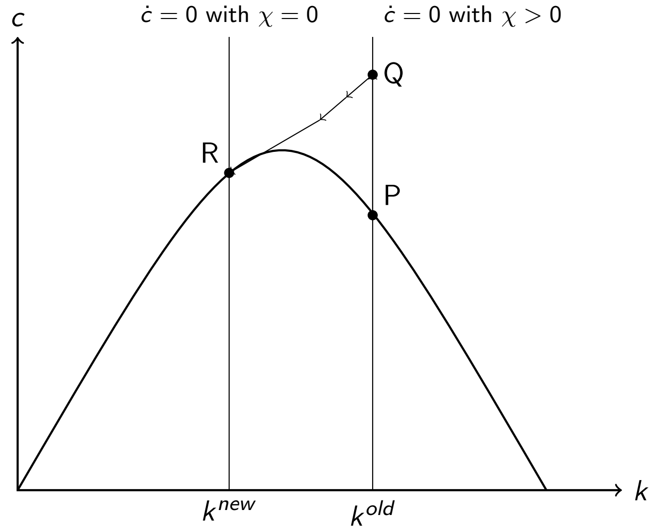
c^*
Initial
Case 1
Case 2
Case 3
c^{gold}
k^*
k^*
k^{*1}
k^{gold}
k^{*2}
k^{*3}

Comparative dynamics

- Suppose the economy is initially at steady state, but χ unexpectedly removed at $t = 0$
- In continuous-time model, how does the economy transition?
- Initial equilibrium system

$$\begin{aligned} \frac{\dot{c}}{c} &= \frac{1}{\theta}\left[\frac{f'(k)}{1-\chi} - \delta - \rho\right] \\ \dot{k} &= f(k) - \delta k - c \end{aligned}$$

- With $\chi \to 0$, $\dot{c} = 0$ locus shifts to the left
- Need initial "jump" to stable arm of new equilibrium system with $\chi = 0$



c
$\dot{c} = 0$ with $\chi = 0$
$\dot{c} = 0$ with $\chi > 0$
Q
R
P
k
k^{new}
k^{old}

Stochastic Growth with Brock-Mirman

Same model with random TFP

- Production
$$\frac{Y_t}{L} = y = f(A_t, k_t)$$
- Let $A_t \in \mathcal{A} = \{A^{(0)}, \ldots, A^{(N)}\}$ be some discrete-valued, $N+1$ state Markov chain
  - Ordered: $f(A^{(j+1)}, k) \geq f(A^{(j)}, k)$ for $0 \leq j \leq N-1$ and all k
  - e.g., $f(A, k) = A \cdot \tilde{f}(k)$

What are we modeling here?

- Brock and Mirman:

  We introduce . . . a random element in the production function. This random variable might also be thought of as an observation error on the capital stock

  Kind of behavioral?

- Kydland and Prescott (*ECMA*, 1982): the business cycle?

A New Problem

- Claim: new optimal growth problem is

$$V(k, A) = \max_{k_{next} \in [0, \bar{k}]} u(f(A, k) + (1 - \delta) k - k_{next}) + \beta \mathbb{E}[V(k_{next}, A_{next})]$$

New features:

- k depends on current A
- Need to average over possible A in next period

But easy to show: problem is still concave and compact

Characterizing the solution

- Appeal to 14.451 logic to figure out that
  - $V(k, z)$ exists, is unique, is concave in k
  - After establishing previous, optimal policy function

    $$k_{next} =: g(k, A)$$

    exists and is strictly increasing in both arguments

- Some hope for answering our original question, which was (roughly):

  When are we sure something sensible will happen in the limit?

- We already know how to "simulate" from this model. Draw sequence $(A_0, A_1, \ldots)$ then calculate

  $$k_1 = g(k_0, A_0), k_2 = g(k_1, A_1), \ldots$$

  but not when this has interpretable long-run behavior

“Results”

- Brock and Mirman is in *JET* so they figure this out elegantly
- Perhaps easier, for us, to specialize to a celebrated simple case:

$$y = Ak^{\alpha} \qquad \delta = 1$$

  in which case policy is

$$k_{next} = g(k, A) = A \cdot \alpha\beta \cdot k^{\alpha}$$

- and phrase the result roughly:

  When A_t doesn't move around like crazy, neither will k in the limit

- Methodologically: can also think about random sequences of taxes, population draws, depreciation rates, . . . , as additional state variables in DP formulation

Optimal Control Practice: q-Theory

Setting

- A price-taking firm is trying to maximize the PDV of its profits
- Big twist from baseline NGM: adjustment costs
- The firm is subject to adjustment costs $\phi(I(t))$ when it changes its capital stock $K(t)$

$$\max_{[K(t),I(t)]_{t=0}^{\infty}} \int_0^{\infty} \exp(-rt)\left[f(K(t)) - I(t) - \phi(I(t))\right]dt$$

subject to

$$\dot{K}_t = I(t) - \delta K(t), \quad K(t) \geq 0$$

- $\phi(I)$: strictly increasing, continuously differentiable, strictly convex

Hamiltonian

Firm's Hamiltonian

$$\hat{H}(K, I, q) \equiv [f(K(t)) - I(t) - \phi(I(t))] + q(t)[I(t) - \delta K(t)]$$

- $q(t)$: costate variable ($\mu(t)$ before)

Necessary conditions for an interior solution?[1]

$$\hat{H}_I(K, I, q) = -1 - \phi'(I(t)) + q(t) = 0 \quad (1)$$

$$\hat{H}_K(K, I, q) = f'(K(t)) - \delta q(t) = rq(t) - \dot{q}(t) \quad (2)$$

$$\lim_{t \to \infty} \exp(-rt)\, q(t) K(t) = 0 \quad (3)$$

[1] Sufficiency is easy to show. See p270 of Ch.7 of Daron's textbook

Roles of Adjustment Cost

Equation (1) implies

$$q(t) = 1 + \phi'(I(t)) \quad \Longrightarrow \quad \dot{q}(t) = \phi''(I(t))\dot{I}(t)$$

Substituting this into Equation (2), we have the law of motion for $I(t)$

$$\dot{I}(t) = \frac{1}{\phi''(I(t))}[(r+\delta)(1+\phi'(I(t))) - f'(K(t))]$$

Intuition for $\phi''(I(t))$?

- If $\phi''(I(t)) = 0$ (close to linear adj. cost), investment jumps (no smoothing)
- If $\phi''(I(t)) > 0$ (convex cost), investment adjustment is slow

Dynamics in Equations

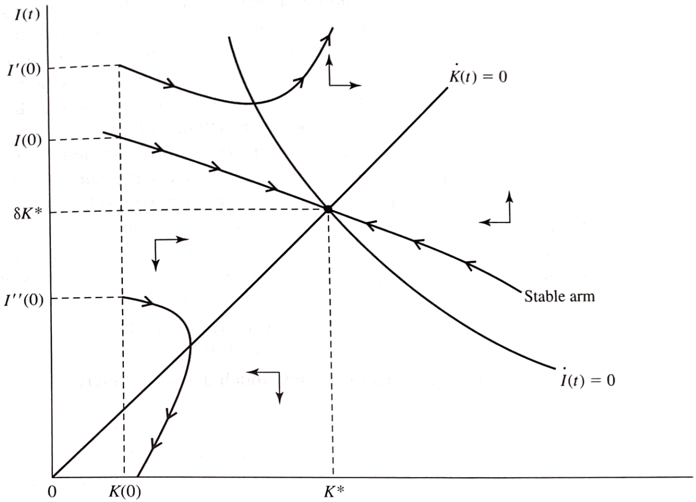
Two ODEs

$$\dot{K}_t = I(t) - \delta K(t), \quad K(t) \geq 0, \quad \text{some } K(0) > 0$$
$$\dot{I}(t) = \frac{1}{\phi''(I(t))}[(r+\delta)(1+\phi'(I(t))) - f'(K(t))]$$

Steady state?

$$I^* = \delta K^*$$
$$f'(K^*) = (r+\delta)(1+\phi'(\delta K^*))$$

Dynamics in Phase Diagram



Tobin's q

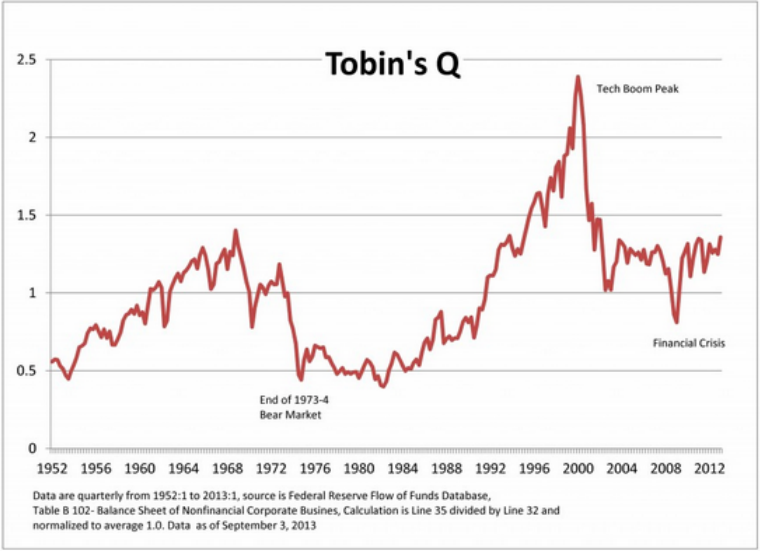
- James Tobin said that the measure of value of the investment to the firm is
  - the value of an extra unit of capital to the firm *divided by* its replacement cost
  - ... when the ratio is high, the firm wants to investment more
  - ... in the steady state, the ratio is 1 or close to 1
- In the model, (marginal) Tobin's q is

$$q(t) = V'(K(t)) \tag{4}$$

- In the steady state

$$q^* = \frac{f'(K^*)}{r+\delta} = 1 + \phi'(\delta K) \tag{5}$$

Tobin's q in the data



[2]

[2] Source: Yahoo Finance

Comments

Issues

- Marginal q (model) v.s. average q (in practice)
- Does not contain important information (irreversibilities etc)

Some papers

- Hayashi, F. (1982). Tobin's marginal q and average q: A neoclassical interpretation. Econometrica: Journal of the Econometric Society, 213-224.
- Lang, L. H., Stulz, R., & Walkling, R. A. (1989). Managerial performance, Tobin's Q, and the gains from successful tender offers. Journal of financial Economics, 24(1), 137-154.
- Andrei, D., Mann, W., & Moyen, N. (2019). Why did the q theory of investment start working?. Journal of Financial Economics, 133(2), 251-272.

Questions?

Appendix

Lifetime budget as Arrow-Debreu

back

- To see this, take $a_{t+1} = (1 + r_t)\, a_t + w_t - c_t$ and multiply both sides by $\left(\prod_{s=0}^{t} \frac{1}{(1+r_s)}\right)$ and sum over $\sum_{t=0}^{T-1}$ to get:

$$
\begin{aligned}
&\sum_{t=0}^{T-1} \left(\prod_{s=0}^{t} \frac{1}{(1 + r_s)} \right) a_{t+1} \\
&= \sum_{t=0}^{T-1} \left(\prod_{s=0}^{t} \frac{1}{(1 + r_s)} \right) (1 + r_t)\, a_t + \sum_{t=0}^{T-1} \left(\prod_{s=0}^{t} \frac{1}{(1 + r_s)} \right) [w_t - c_t]
\end{aligned}
$$

Lifetime budget as Arrow-Debreu

- Now, note that

$$\sum_{t=0}^{T-1}\left(\prod_{s=0}^{t}\frac{1}{(1+r_s)}\right)a_{t+1}-\sum_{t=0}^{T-1}\left(\prod_{s=0}^{t}\frac{1}{(1+r_s)}\right)(1+r_t)\,a_t$$
$$=\prod_{s=0}^{T-1}\frac{1}{(1+r_s)}a_T-a_0$$

Lifetime budget as Arrow-Debreu

- Then, going back to our original equation:

$$\sum_{t=0}^{T-1}\left(\prod_{s=0}^{t}\frac{1}{(1+r_s)}\right)[w_t-c_t]=\prod_{s=0}^{T-1}\frac{1}{(1+r_s)}a_T-a_0$$

$$\lim_{T\to\infty}\sum_{t=0}^{T-1}\left(\prod_{s=0}^{t}\frac{1}{(1+r_s)}\right)[w_t-c_t]+a_0=\lim_{T\to\infty}\prod_{s=0}^{T-1}\frac{1}{(1+r_s)}a_T\geq 0$$

$$\sum_{t=0}^{\infty}\left(\prod_{s=0}^{t}\frac{1}{(1+r_s)}\right)w_t+a_0\geq\sum_{t=0}^{\infty}\left(\prod_{s=0}^{t}\frac{1}{(1+r_s)}\right)c_t$$

$$\sum_{t=0}^{\infty}\left(\prod_{s=1}^{t}\frac{1}{(1+r_s)}\right)w_t+(1+r_0)\,a_0\geq\sum_{t=0}^{\infty}\left(\prod_{s=1}^{t}\frac{1}{(1+r_s)}\right)c_t$$

which is exactly the budget constraint in the Arrow-Debreu CE

Lifetime budget as AD

- Note that conceptually:

$$\sum_{t=0}^{\infty}\left(\prod_{s=1}^{t}\frac{1}{(1+r_s)}\right)w_t+(1+r_0)\,a_0\geq\sum_{t=0}^{\infty}\left(\prod_{s=1}^{t}\frac{1}{(1+r_s)}\right)c_t$$

$$(\text{Human Wealth at } t=0)+(\text{Financial Wealth at } t=0)$$
$$\geq(NPV \text{ of Consumption at } t=0)\,.$$

- Since $u(\cdot)$ is strictly inc. we know in eq we will have:

$$\sum_{t=0}^{\infty}\left(\prod_{s=1}^{t}\frac{1}{(1+r_s)}\right)w_t+(1+r_0)\,a_0=\sum_{t=0}^{\infty}\left(\prod_{s=1}^{t}\frac{1}{(1+r_s)}\right)c_t$$

which is also what the TVC of the sequential Eq told us