14.452 Recitation 4: Overlapping Generations

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Recitation Plan: Solve the canoncial overlapping generations model.

1 Setup

Consider the two-period canonical overlapping generations model with log preferences

$$\log(c_1(t)) + \beta \log(c_2(t+1))$$

for each individual. Suppose that there is population growth at the rate n. Individuals work only when they are young, and supply one unit of labor inelastically. The production technology is given by

$$Y(t) = A(t)K(t)^{\alpha} L(t)^{1-\alpha},$$

where $A(t+1) = (1+g)A(t)$, with $A(0) > 0$ and $g > 0$.

2 Solution

Part 1. Define a competitive equilibrium and the steady-state equilibrium.

Solution: A *competitive equilibrium* is an allocation $[c_1(t), c_2(t), K(t)]_{t\geq 0}$ and prices $[R(t), w(t)]_{t\geq 0}$ such that

(i) the consumption values $c_1(t)$ and $c_2(t+1)$ are determined by the solution to generation t's optimization problem, taking $w(t)$ and $R(t+1)$ as given:

$$\max_{c_1(t), c_2(t+1)} \log(c_1(t)) + \beta \log(c_2(t+1)) \quad \text{s.t.} \quad c_1(t) + \frac{c_2(t+1)}{R(t+1)} \leq w(t),$$

and $c_2(0) = R(0)K(0)$;

(ii) prices $R(t)$ and $w(t)$ are given by the representative firm's optimality conditions

$$R(t) = F_K(K(t), L(t)) \quad \text{and} \quad w(t) = F_L(K(t), L(t));$$

(iii) capital accumulates according to $K(t+1) = S(t)$, where $S(t) = L(t)(w(t) - c_1(t))$, with $K(0) > 0$ given.

A *steady-state* equilibrium is a competitive equilibrium in which output $Y(t)$, capital $K(t)$, and total consumption $C(t) = c_1(t)L(t) + c_2(t)L(t-1)$ all grow at constant rates.

Part 2. Can you apply the First Welfare Theorem to this competitive equilibrium?

Solution: Not necessarily – see the discussion below.

Part 3. Characterize the steady-state equilibrium and show that it is globally asymptotically stable.

Solution: Begin by writing generation t's problem as an optimization problem over savings $s(t) = w(t) - c_1(t)$:

$$\max_{s(t) \in [0, w(t)]} \log(w(t) - s(t)) + \beta \log(R(t+1)s(t)).$$

The solution implies that the household saves a constant fraction of its income (wealth), regardless of the interest rate $R(t+1)$:

$$s(t) = \frac{\beta}{1+\beta} w(t).$$

This owes to the assumption of log preferences, which are equivalently Cobb-Douglas preferences over consumption when young and when old (take the exponential of the household's objective function to see this). Aggregating over generation t households to arrive at aggregate savings $S(t)$, we can use the capital accumulation equation to find

$$K(t+1) = S(t) = \frac{\beta}{1+\beta} L(t) w(t).$$

Using the firm's optimality condition, we can substitute for the wage:

$$K(t+1) = \frac{\beta}{1+\beta} (1-\alpha) A(t) K(t)^{\alpha} L(t)^{1-\alpha}$$

Note that with Cobb-Douglas production, the wage bill $L(t)w(t)$ is just a fraction $1-\alpha$ of total output $Y(t)$. The capital accumulation equation above is a *non-autonomous* first-order difference equation in capital: Given $K(t)$, it tells us how to calculate $K(t+1)$, but this calculation is time-varying because both $L(t)$ and $A(t)$ are growing. As in the Solow and neoclassical growth models, to characterize the steady-state we can attempt to write the capital accumulation equation as an *autonomous* first-order difference equation in a "detrended" or "normalized" capital-like variable. Start by dividing both sides by $L(t)$, using the assumption that $L(t+1) = (1+n)L(t)$:

$$\frac{K(t+1)}{L(t+1)} = \frac{1-\alpha}{1+n}\frac{\beta}{1+\beta}A(t)\left(\frac{K(t)}{L(t)}\right)^{\alpha}.$$

This is a first-order difference equation in the capital-labor ratio $K(t)/L(t)$, but it is again non-autonomous when there is technological progress ($g > 0$). We can perform the same "trick" again but with $A(t)$ by writing $A(t) = A(t)^{\frac{1}{1-\alpha}-\frac{\alpha}{1-\alpha}}$ and dividing each side by $A(t)^{\frac{1}{1-\alpha}}$:

$$\frac{K(t+1)}{A(t+1)^{\frac{1}{1-\alpha}}L(t+1)} = \frac{1}{(1+g)^{\frac{1}{1-\alpha}}}\frac{1-\alpha}{1+n}\frac{\beta}{1+\beta}\left(\frac{K(t)}{A(t)^{\frac{1}{1-\alpha}}L(t)}\right)^{\alpha}.$$

Here we also used the assumption that $A(t+1) = (1+g)A(t)$. We then arrive at an *autonomous* first-order difference equation in the capital-effective labor ratio $\tilde{k}(t) = K(t)/A(t)^{\frac{1}{1-\alpha}}L(t)$. Why does the technology shock $A(t)$ enter with the exponent $\frac{1}{1-\alpha}$? With Cobb-Douglas production, a Hicks-neutral shock $A(t)$ has precisely the same effect on the production technology as the labor-augment shock $A(t)^{\frac{1}{1-\alpha}}$. Uzawa's Theorem tells us that in any balanced growth path the capital stock must grow at the same rate as effective labor, which in this model is given by $A(t)^{\frac{1}{1-\alpha}}L(t)$. So it makes sense to choose this as a normalizing variable for the capital stock – and we know that it is the *right* choice because the capital accumulation equation becomes autonomous (i.e., stationary) when we do this.

To characterize the steady-state, we write the difference equation in terms of $\tilde{k}(t)$:

$$\tilde{k}(t+1) = \frac{1}{(1+g)^{\frac{1}{1-\alpha}}}\frac{1}{1+n}\frac{\beta}{1+\beta}(1-\alpha)\tilde{k}(t)^{\alpha}.$$

Any steady-state $\tilde{k}^*$ must satisfy this equation with $\tilde{k}(t+1) = \tilde{k}(t) = \tilde{k}^*$. The unique non-zero steady state is then

$$\tilde{k}^* = \left[\frac{1}{(1+g)^{\frac{1}{1-\alpha}}}\frac{1-\alpha}{1+n}\frac{\beta}{1+\beta}\right]^{\frac{1}{1-\alpha}}.$$

In this steady state, capital, output, and total consumption all grow at the same constant rate:

$$
\begin{aligned}
K(t) &= \tilde{k}^* A(t)^{\frac{1}{1-\alpha}} L(t) \\
&= (1+g)^{\frac{1}{1-\alpha}}(1+n)K(t-1), \\
Y(t) &= \left(\tilde{k}^*\right)^{\alpha} A(t)^{\frac{1}{1-\alpha}} L(t) \\
&= (1+g)^{\frac{1}{1-\alpha}}(1+n)Y(t-1), \\
C(t) &= Y(t) - K(t+1) \\
&= Y(t) - (1+g)^{\frac{1}{1-\alpha}}(1+n)K(t) \\
&= \left[1-(1+g)^{\frac{1}{1-\alpha}}(1+n)\left(\tilde{k}^*\right)^{1-\alpha}\right] Y(t).
\end{aligned}
$$

In particular, all "per capita" variables $(Y(t)/L(t),\, K(t)/L(t),\, \ldots)$ grow at the constant rate $(1+g)^{\frac{1}{1-\alpha}}$. The wage similarly grows at the rate $(1+g)^{\frac{1}{1-\alpha}}$, and the interest rate is constant:

$$
\begin{aligned}
w(t) &= (1-\alpha)A(t)\left(\frac{K(t)}{L(t)}\right)^{\alpha} &&= (1-\alpha)\left(\tilde{k}^*\right)^{\alpha} A(t)^{\frac{1}{1-\alpha}}, \\
R(t) &= \alpha A(t)\left(\frac{L(t)}{K(t)}\right)^{1-\alpha} &&= \alpha\left(\tilde{k}^*\right)^{-(1-\alpha)}
\end{aligned}
$$

The steady state is globally stable provided that $\tilde{k}(t) \to \tilde{k}^*$ given any starting value $\tilde{k}(0)$. To show this, we can prove the stronger result that $\tilde{k}(t)$ converges *monotonically* to $\tilde{k}^*$: If $\tilde{k}(0) < \tilde{k}^*$, $\tilde{k}(t)$ increases at each t and converges to $\tilde{k}^*$, but if $\tilde{k}(0) > \tilde{k}^*$, $\tilde{k}(t)$ decreases at each t and converges to $\tilde{k}^*$. I prove this just when $\tilde{k}(0) < \tilde{k}^*$, because the argument is identical for the case with high initial capital. Write the difference equation for $\tilde{k}$ as

$$
\tilde{k}(t+1) = G\left(\tilde{k}(t)\right), \quad \text{where} \quad G\left(\tilde{k}\right) = \frac{1}{(1+g)^{\frac{1}{1-\alpha}}}\frac{1}{1+n}\frac{\beta}{1+\beta}(1-\alpha)\tilde{k}^{\alpha}.
$$

The function G is strictly increasing, and $\tilde{k}^*$ is the unique non-zero solution to the equation $\tilde{k} = G\left(\tilde{k}\right)$. With $\tilde{k}(t) < \tilde{k}^*$, these facts immediately imply

$$
\tilde{k}(t+1) = G\left(\tilde{k}(t)\right) < G\left(\tilde{k}^*\right) = \tilde{k}^*.
$$

Hence the capital-labor ratio is always bounded above by the steady-state value $\tilde{k}^*$. Moreover, the capital-labor ratio $\tilde{k}(t)$ is increasing over time:

$$
\tilde{k}(t+1) > \tilde{k}(t) \iff G\left(\tilde{k}(t)\right) > \tilde{k}(t) \iff \tilde{k}(t) < \tilde{k}^*.
$$

The final implication holds by direct calculation. Since $\tilde{k}(t)$ is strictly increasing and bounded above by the unique fixed point $\tilde{k}^*$ of G, we conclude that $\tilde{k}(t) \uparrow \tilde{k}^*$. Repeating the same argument for $\tilde{k}(0) > \tilde{k}^*$, we conclude that the steady-state equilibrium is globally stable.

Part 4. What is the effect of an increase in g on the equilibrium path?

Solution: An increase in g raises output per capita, the capital-labor ratio, consumption per capital, the wage, and the interest rate at each time t – not just in the steady state. To see this, recall from above the capital-labor ratio statisfies the non-autonomous first-order difference equation

$$\frac{K(t+1)}{L(t+1)} = \frac{1-\alpha}{1+n}\frac{\beta}{1+\beta}A(t)\left(\frac{K(t)}{L(t)}\right)^{\alpha}.$$

An increase in g raises $A(t)$ at each time $t > 0$. Since $K(0)/L(0)$ is fixed and the right side of this equation is increasing in $K(t)/L(t)$ and $A(t)$, we immediately observe that an increase in g raises $K(t)/L(t)$ at each time $t > 0$. This immediately implies the corresponding result for output per capita when we note

$$\frac{Y(t)}{L(t)} = A(t)\left(\frac{K(t)}{L(t)}\right)^{\alpha}.$$

The wage and interest rate satisfy

$$w(t) = (1-\alpha)A(t)\left(\frac{K(t)}{L(t)}\right)^{\alpha},$$

$$R(t) = \alpha\left(\frac{A(t)^{\frac{1}{1-\alpha}}L(t)}{K(t)}\right)^{1-\alpha}.$$

The same argument as above implies that $w(t)$ is increasing in g at each time $t > 0$. To determine the comparative static for the interest rate $R(t)$, recall that the capital-effective labor ratio satisfies the autonomous first-order difference equation

$$\tilde{k}(t+1) = \frac{1}{(1+g)^{\frac{1}{1-\alpha}}}\frac{1}{1+n}\frac{\beta}{1+\beta}(1-\alpha)\tilde{k}(t)^{\alpha}.$$

The right side of this equation is decreasing in g and increasing in $\tilde{k}(t)$, and the initial value $\tilde{k}(0)$ is fixed. These facts immediately imply that $\tilde{k}(t)$ is decreasing in g at each time $t > 0$. But since the interest rate is given by $R(t) = \alpha\tilde{k}(t)^{-(1-\alpha)}$, we observe that the interest rate at each time $t > 0$ is increasing in g. Finally, note that since both the wage $w(t)$ and the interest rate $R(t+1)$ are increasing in g, consumption for generation t while young $c_1(t)$ and while old $c_2(t+1)$ must both be increasing in g.

Part 5. In the rest of the question, assume that $g = 0$. Suppose that the equilibrium involves $r^* < n$. Explain why the equilibrium is referred to as "dynamically inefficient" in this case. Show that an unfunded Social Security system can increase the welfare of *all* future generations.

Solution: When $r^* = R^* - 1 < n$, the equilibrium is "dynamically inefficient" because the steady-state or limiting capital stock exceeds the golden rule capital stock that maximizes steady-state consumption. Intuitively, the equilibrium overaccumulates capital when $r^* < n$, and reducing the capital stock (equivalently, the quantity of savings) at each date would allow for an increase in consumption at each date. This implies that the equilibrium is *not Pareto optimal*, and we will provide a constructive proof to show that there are alternative allocations that strictly increase the welfare of all generations.

We proceed by introducing an unfunded Social Security system to the economy, which consists of a tax d on each generation while young and a corresponding transfer $(1+n)d$ to each generation while old. Since the population of each generation grows at rate n, this amounts to a mandatory transfer from young to old at each time t. I will show that for d sufficiently small, the equilibrium with the Social Security system Pareto dominates the equilibrium without the Social Security system.

The optimization problem for generation t now becomes

$$\max_{s(t)\in[0,w(t)-d]} \log\left(w(t)-d-s(t)\right)+\beta\log\left(R(t+1)s(t)+(1+n)d\right).$$

The interior first-order condition must be satisfied for d sufficiently small:

$$\frac{1}{w(t)-d-s(t)}=\frac{\beta R(t+1)}{R(t+1)s(t)+(1+n)d}.$$

Savings by generation t are then

$$s(t)=\frac{\beta}{1+\beta}w(t)-\frac{1}{1+\beta}\left(\beta+\frac{1+n}{R(t+1)}\right)d.$$

Note that with fixed prices $w(t)$ and $R(t+1)$, savings are always smaller when $d > 0$ as each household attempts to compensate for the lower consumption at t and the higher consumption at $t+1$ effected by the Social Security system. (But we have to see if this remains true in *general equilibrium* after prices adjust.) Aggregating across generation t households and using the representative firm's optimality conditions to substitute for the prices $w(t)$ and $R(t+1)$, we find that capital satisfies the non-autonomous first-order difference equation

$$K(t+1) = \frac{\beta}{1+\beta}(1-\alpha)K(t)^{\alpha}L(t)^{1-\alpha} - \frac{1}{1+\beta}\left[\beta + \frac{1+n}{\alpha}\left(\frac{K(t+1)}{L(t+1)}\right)^{1-\alpha}\right]dL(t)$$

Letting $k(t) = K(t)/L(t)$ denote the capital-labor ratio, we can divide through by $L(t)$ and rearrange to find the autonomous first-order difference equation

$$(1+n)(1+\beta)k(t+1) + \left[\beta + \frac{1+n}{\alpha}k(t+1)^{1-\alpha}\right]d = \beta(1-\alpha)k(t)^{\alpha}$$

The second term on the left side is the new term that arises with $d > 0$. Just as in the standard OLG model, this difference equation fully characterizes the equilibrium with the Social Security system: All quantities and prices at each time can be written as a function of the capital-labor ratio $k(t)$ and the (exogenous) number of workers $L(t)$. However, when $d > 0$ we cannot generally solve for $k(t+1)$ as a function of $k(t)$ in closed form. But since the left side of this equation is strictly increasing in $k(t+1)$, tends to zero as $k(t+1) \to 0$, and tends to infinity as $k(t+1) \to \infty$, there is still a unique solution $k(t+1)$. To perform comparative statics with respect to d, we write $k(t,d)$ to emphasize the dependence of the sequence of capital-labor ratios on d, and we write the difference equation in more compact form as

$$H(k(t+1,d),d) = \beta(1-\alpha)k(t,d)^{\alpha}.$$

We can implicitly differentiate with respect to d to find

$$k_d(t+1,d) = \frac{\beta(1-\alpha)\alpha k(t,d)^{\alpha-1}k_d(t,d) - H_d(k(t+1,d),d)}{H_k(k(t+1,d),d)}.$$

I claim that this equation implies $\partial k(t+1,d)/\partial d < 0$. Since $H_d > 0$ and $H_k > 0$, this holds provided that $k_d(t,d) < 0$. But at $t = 0$, since $k(0,d) = k(0)$ is exogenously fixed, this equation reduces to

$$k_d(1,d) = -\frac{H_d(k(1,d),d)}{H_k(k(1,d),d)} < 0.$$

By induction, we can conclude that $k_d(t+1,d) < 0$ for $t \geq 0$.

Thus far, we have established that expanding the Social Security system (i.e., raising d) lowers the capital-labor ratio at each time in the (unique) competitive equilibrium. How does this fact help us show that we can achieve a Pareto improvement by introducing the Social Security system? The idea is that a small increase from $d = 0$ to $d > 0$ reduces the capital-labor ratio at each time, and since the economy overaccumulates capital in the $d = 0$ equilibrium, this adjustment could (and in fact will!) raise consumption at each time by reducing the overaccumulation. For our purposes, it will suffice to show that each generation's budget constraint slackens when we raise d marginally from $d = 0$ to $d > 0$. For arbitrary $d > 0$, generation t's budget constraint is

$$c_1(t) \leq w(t,d) + \left(\frac{1+n}{R(t+1,d)} - 1\right)d - \frac{c_2(t+1)}{R(t+1,d)}.$$

It suffices to show that, when $c_1(t)$ and $c_2(t+1)$ take their steady-state equilibrium values with $d = 0$, the right side of this inequality is increasing in d near $d = 0$. Recall that these steady-state equilibrium values are given by

$$c_1^* = w^* - s^* = w^* - (1+n)\,k^*,$$
$$c_2^* = R^*\,(1+n)\,k^*.$$

The right-hand side (RHS) of the inequality above can then be written

$$\text{RHS}(t,d) = w(t,d) + \left(\frac{1+n}{R(t+1,d)} - 1\right)d - \frac{R^*\,(1+n)\,k^*}{R(t+1,d)}$$

Differentiating with respect to d and evaluating at $d = 0$, we have

$$\text{RHS}_d(t,0) = w_d(t,0) + \frac{1+n}{R^*} - 1 + \frac{(1+n)\,k^*}{R^*} R_d(t+1,0),$$

where we used the identity $R(t+1,0) = R^*$. To calculate the derivatives $w_d(t,0)$ and $R_d(t+1,0)$, we make use of the equilibrium price conditions

$$w(t,d) = (1-\alpha)\,k(t,d)^{\alpha},$$
$$R(t+1,d) = \alpha k(t+1,d)^{-(1-\alpha)}.$$

Differentiating yields

$$w_d(t,d) = \alpha\,(1-\alpha)\,k(t,d)^{-(1-\alpha)} k_d(t,d),$$
$$R_d(t+1,d) = -\alpha\,(1-\alpha)\,k(t+1,d)^{-(2-\alpha)} k_d(t+1,d).$$

We can then write

$$\text{RHS}_d(t,0) = \frac{1+n}{R^*} - 1 + \alpha(1-\alpha)(k^*)^{-(1-\alpha)}\left[k_d(t,0) - \frac{1+n}{R^*}k_d(t+1,0)\right].$$

Our final observation is that the evolution equation for $k_d(t+1,0)$ implies that $k_d(t+1,0) < k_d(t,0)$ for $t \geq 0$. Intuitively, starting from the $d = 0$ steady-state capital-labor ratio k^*, introducing the unfunded Social Security system $d > 0$ requires a reduction in the capital-labor ratio at each time to reach the new and lower steady-state capital-labor ratio $k^*(d)$. The capital-labor ratio declines monotonically to $k^*(d)$, so in response to the introduction of $d > 0$, the capital-labor ratio at $t+1$ must fall more than at t relative to the initial steady-state k^*. This observation implies

$$\begin{aligned}\text{RHS}_d(t,0) &> \frac{1+n}{R^*} - 1 + \alpha(1-\alpha)(k^*)^{-(1-\alpha)}\left[k_d(t+1,0) - \frac{1+n}{R^*}k_d(t+1,0)\right] \\ &= \left(1 - \alpha(1-\alpha)(k^*)^{-(1-\alpha)}k_d(t+1,0)\right)\left(\frac{1+n}{R^*} - 1\right)\end{aligned}$$

The first factor is positive since $k_d(t+1,0) < 0$, and the second factor is positive since the initial steady-state equilibrium is dynamically inefficient $(1+n > R^*)$. Hence $\text{RHS}_d(t,0) > 0$, and since preferences are locally non-satiated, introducing a (small) unfunded Social Security system must strictly raise welfare for each generation in equilibrium.[1]

Part 6. Show that if $r^* > n$, then any unfunded Social Security system that increases the welfare of the current old generation must reduce the welfare of some future generation.

Solution: In this case, we can directly apply the First Welfare Theorem with a countably infinite number of households (see problem set solutions for details).

[1]Technically I've shown this only for generations $t \geq 0$, but it is immediate that generation $t = -1$ (i.e., the initial elderly who own the initial capital stock) strictly benefits from the new transfer of $(1+n)d$ per capita.