# Recitation 2: Diamond-Mirrlees I

Todd Lensman

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**Recitation Plan:** Review the Diamond-Mirrlees production efficiency result and applications

## 1 General Model with a Representative Consumer

**Consumption.** The economy has a continuum of identical consumers. Each consumer has preferences defined over her own net consumption vector $x \in X$, where $X \subseteq \mathbb{R}^n$ is assumed convex with a non-empty interior. These preferences are represented by a utility function $u(x)$, assumed differentiable, concave, and locally non-satiated.

Given consumer prices $q \in \mathbb{R}^n$ and a lump-sum tax $T \in \mathbb{R}$, the representative consumer solves

$$\max_{x \in X} u(x) \quad \text{subject to} \quad q \cdot x + T \leq 0. \tag{1.1}$$

Throughout, we assume that the solution occurs at an interior point of $X$, so the budget constraint is the only active constraint.

**Private Production.** The private sector's production technology is described by the transformation function $F : \mathbb{R}^n \to \mathbb{R}$, where a net output vector is feasible if and only if $F(y) \leq 0$. We assume that $F$ is differentiable and homogeneous of degree one, so that the production technology has constant returns to scale.

Given the transformation function $F$ and producer prices $p$, the representative competitive firm maximizes profits over production vectors $y$:

$$\max_{y \in \mathbb{R}^n} p \cdot y \quad \text{subject to} \quad F(y) \leq 0. \tag{1.2}$$

**Government.** The government uses commodity taxes and a lump-sum tax to finance a vector of public spending $g \in \mathbb{R}^n$, and it can also engage in public production of commodities. Let $z \in \mathbb{R}^n$ denote the vector of net government production, inclusive of spending $g$, and let $G : \mathbb{R}^n \to \mathbb{R}$

denote the government's transformation function. The government's production constraint is $G(z) \leq 0$, and its implied budget constraint is

$$(q-p)\cdot x + T + p\cdot z = 0. \tag{1.3}$$

**Equilibrium.** An equilibrium is this economy is essentially a Walrasian competitive equilibrium with taxes: a tuple $(x, z, q, p, T)$ such that

(i) the government's budget constraint is satisfied;

(ii) the government's production constraint is satisfied;

(iii) the representative consumer chooses consumption vector $x$ given $q$ and $T$;

(iv) the representative firm chooses production vector $y$ given $p$; and

(v) all markets clear, $x = y + z$.

# 2 Production Efficiency + Applications

## 2.1 The Result

Toward the production efficiency result of Diamond and Mirrlees (1971a), we restrict the government's policy tools by setting $T = 0$. Otherwise, the government can optimally finance any (net) government spending by levying a lump-sum tax on the representative consumer and leaving all relative prices undistorted – this is a direct implication of the First Welfare Theorem. In this case, aggregate production efficiency (i.e., $F(y) = 0$ and $G(z) = 0$) is necessarily satisfied at the optimum.

With $T = 0$, the welfarist government's problem is as follows:

$$\max_{z,q,p} u(x(q)) \quad \text{subject to} \quad G(z) \leq 0, \tag{2.1}$$

$$x(q) = y(p) + z. \tag{2.2}$$

Here $x(q)$ denotes the solution to the consumer's problem given $q$ and $T = 0$, and $y(p)$ denotes the solution to the firm's problem given $p$. Note that since the consumer's budget constraint is necessarily satisfied with equality and the firm makes zero profits, the market-clearing conditions in the government's problem immediately imply that the government's budget constraint is satisfied with $T = 0$.

To simplify this problem, we observe that since the firm's problem is concave, the solution is characterized by the first-order conditions

$$\frac{p_i}{p_1} = \frac{F_i}{F_1} \quad i \in \{1, \ldots, n\}. \tag{2.3}$$

Equilibrium with $p \neq 0$ also implies that the firm must be productively efficient, $F(y) = 0$. As we have seen in lecture, a key implication of this fact is that the government can obtain any feasible and productively efficient net output vector $y$ from the private sector with an appropriate choice of producer prices $p$. We can then equivalently write the government's problem without producer prices $p$ but with a government choice of private sector net output $y$:

$$\max_{y,z,q} u(x(q)) \quad \text{subject to} \quad F(y) \leq 0, \tag{2.4}$$

$$G(z) \leq 0, \tag{2.5}$$

$$x(q) = y + z. \tag{2.6}$$

Note that I have already incorporated a relaxation by allowing the government to choose a productively inefficient private net output vector (with $F(y) < 0$). The following result demonstrates that this relaxation does not change the solution to the government's problem:

**Theorem 2.1** (Productive Efficiency)**.** Let $(y, z, q)$ be a solution to the government's problem with $x_k(q) \neq 0$. Then

$$F(y) = 0, \qquad G(z) = 0, \qquad \frac{F_i}{F_1} = \frac{G_i}{G_1} \quad i \in \{1, \ldots, n\}. \tag{2.7}$$

*Proof.* Let $\lambda^F$ and $\lambda^G$ be the multipliers on the private and public production constraints, and let $\gamma_i$ be the multiplier on the market-clearing constraint for commodity $i$. Then the first-order conditions with respect to $y_i$ and $z_i$ are

$$\lambda^F F_i = \gamma_i = \lambda^G G_i. \tag{2.8}$$

Provided that the multipliers $\lambda^F$ and $\lambda^G$ are non-zero, and that the transformation functions $F$ and $G$ are strictly monotone in some commodity, we can conclude. To establish these facts, note that the first-order condition with respect to $q_k$ can be written

$$\alpha x_k(q) = \sum_{i=1}^{n} \gamma_i \frac{\partial x_i}{\partial q_k}. \tag{2.9}$$

where $\alpha > 0$ denotes the consumer's marginal utility of wealth. Hence there exists some commodity, say 1, such that $\gamma_1 \neq 0$. This implies $\lambda^F, \lambda^G, F_1, G_1 \neq 0$. ∎

**Remark.** In the version of the model with heterogeneous consumers, a nearly identical argument demonstrates that productive efficiency must hold at the optimum. Except for technical conditions that ensure the Walrasian equilibrium is well-behaved, the only modification we must make to our assumptions is that there exists some commodity $k$ such that $x_k^h(q) \geq 0$ for all households $h$ or $x_k^h(q) \leq 0$ for all households $h$, with a strict inequality for some $h$. This implies that the multiplier $\gamma_k$ is again nonzero. Try re-doing the proof with heterogeneous consumers to see how this works, or check out Section IV of Diamond & Mirrlees (1971a) for full technical details. Alternatively, with heterogeneous consumers, the argument becomes even easier if we allow a uniform lump-sum tax/subsidy (uniformity implies that the optimum may still be second-best).

## 2.2 Applications

### Management of Public Production

What objective should be assigned to a publicly-owned firm when the government seeks to maximize welfare over commodity taxes and the public production vector? The production efficiency theorem immediately implies that public and private marginal technical rates of substitution should be equated, so the publicly-owned firm should act exactly like a privately-owned firm (but with technology $G$ instead of $F$).

### Intermediate Taxation

Should we ever tax firm-to-firm transactions? Not according to the production efficiency result! To see this, we can reinterpret $F$ and $G$ as the production technologies available in two different private sectors (e.g., manufacturing and services). Suppose, as is necessary for intermediate good taxation, that the government can assign different price vectors $p^F$ and $p^G$ to sectors $F$ and $G$, respectively. Then by the same argument as above, the government can use prices $p^F$ and $p^G$ to obtain any productively efficient net output vectors $y$ and $z$ from sectors $F$ and $G$. Relaxing the requirement of within-sector productive efficiency, we obtain the same optimization problem for the government that we analyzed in the proof of the production efficiency result. The conclusion: each sector should be internally productively efficient, and we should equate marginal rates of technical substitution across sectors. To see how these facts imply aggregate productive efficiency, we can analyze the aggregate production technology and characterize its frontier. In particular, let $a = y + z$ denote aggregate net output, and note that the aggregate technology is described by the transformation function

$$A(a) \quad := \quad a_1 - \max_{y,z} y_1 + z_1 \tag{2.10}$$

$$\text{subject to} \tag{2.11}$$

$$y_i + z_i \geq a_i \quad i \in \{2, \ldots, n\}, \tag{2.12}$$

$$F(y) \leq 0, \tag{2.13}$$

$$G(z) \leq 0. \tag{2.14}$$

The optimization problem is convex, so the solution is characterized by the first-order conditions. In particular, we must have

$$1 = \lambda^F F_1 = \lambda^G G_1, \tag{2.15}$$

$$\gamma_i = \lambda^F F_i = \lambda^G G_i \quad i \in \{2, \ldots, n\}. \tag{2.16}$$

Under weak monotonicity conditions on the production technologies,[1] the Lagrange multipliers $\lambda^F$ and $\lambda^G$ are non-zero, and the marginal technical rates of substitution are equated across sectors. The aggregate production frontier is then characterized by the conditions

$$F(y) = 0, \qquad G(z) = 0, \qquad \frac{F_i}{F_k} = \frac{G_i}{G_k} \quad i \in \{1, \ldots, n\}. \tag{2.17}$$

But these are precisely the conditions implied by the production efficiency theorem! Thus the production efficiency theorem implies productive efficiency in each private sector and equality of marginal technical rates of substitution across sectors (i.e., no intermediate taxation across sectors), which in turn are equivalent to aggregate production efficiency.

**Trade**

Should a domestic welfare-maximizing government ever tax or subsidize imports or exports? The production efficiency theorem again says no, assuming the government acts as a price-taking firm on the world market. To see this, suppose that commodities 1 and 2 are tradable,

[1] These essentially require that "you can't get something for nothing."

and suppose the home country faces international prices $p_1^I$ and $p_2^I$. From the perspective of the home country, trade amounts to a private production technology that allows the exchange of commodity 1 for commodity 2 at marginal technical rate of substitution $p_1^I/p_2^I$. Formally, we can reinterpret technology $G$ as that of a private importer-exporter:

$$G(z_1, z_2) = p_1^I z_1 + p_2^I z_2. \tag{2.18}$$

Proceeding according to the proof of the production efficiency theorem, we can conclude that the solution must feature

$$\frac{G_1}{G_2} = \frac{p_1^I}{p_2^I} = \frac{F_1}{F_2}. \tag{2.19}$$

In particular, the government should not distort marginal technical rates of substitution away from those that prevail at international prices. For example, suppose that commodity 1 is produced by domestic firms using commodity 2 as an input. The government should not subsidize domestic output of commodity 1, because the last unit of commodity 2 used in domestic production could be more efficiently used by exchanging it for commodity 1 on the world market. The "return" in units of commodity 1 from international trade is more favorable because domestic firms over-produce commodity 1 due to the subsidy, leading to diminished marginal returns in production.