Recitation 1: Pigouvian Taxation

Todd Lensman

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Recitation Plan: Discuss and solve examples of the general Pigouvian taxation model

1 General Model with a Representative Consumer

Consumption. The economy has a continuum of identical consumers. Each consumer has preferences defined over her own net consumption vector $x \in X$ and the average net consumption vector in the population $\bar{x} \in X$, where $X \subseteq \mathbb{R}^n$ is assumed convex with a non-empty interior. These preferences are represented by a utility function $u(x, \bar{x})$, assumed differentiable in all arguments and concave and locally non-satiated in x for any $\bar{x}$. Since $\bar{x}$ directly enters u, we have the possibility of consumption externalities – one consumer's choice can directly affect another's utility.

Given consumer prices $q \in \mathbb{R}^n$, a lump-sum tax $T \in \mathbb{R}$, and the average consumption choice $\bar{x}$, the representative consumer solves

$$\max_{x \in X} u(x, \bar{x}) \quad \text{subject to} \quad q \cdot x + T \leq 0. \tag{1.1}$$

Throughout, we assume that the solution occurs at an interior point of X, so the budget constraint is the only active constraint.

Example 1.1. Consider an economy that only has two "goods," consumption c and labor l. The representative consumer has endowments $\tilde{c}$ and $\tilde{l}$ of each, and we naively define the consumer's preferences by the uility function $v(c, l)$, where v is concave, strictly increasing in c, and strictly decreasing in l (note that there are no externalities). Suppose we also impose that final consumption must be non-negative, while labor supply must be non-negative and weakly smaller than the consumer's labor endowment. The consumer's problem is then

$$\max_{c \geq 0, \tilde{l} \geq l \geq 0} v(c, l) \quad \text{subject to} \quad q_c(c - \tilde{c}) + T \leq q_l l. \tag{1.2}$$

To cast this specification of consumption in terms of the general model, we recall that x is interpreted as the vector of transacted quantities in the market. That is, $x_c = c - \tilde{c}$ and $x_l = -l$, where x_l is negative to maintain the convention that prices are non-negative. Then the consumption set is $X := \left\{x \mid x_c \geq -\tilde{c}, 0 \geq x_l \geq -\tilde{l}\right\}$, and the consumer's utility function over x is $u(x) := v(x_c + \tilde{c}, -x_l)$.

Production. The economy's production technology is described by the transformation function $F : \mathbb{R}^n \to \mathbb{R}$, where a net output vector is feasible if and only if $F(y) \leq 0$. We assume that F is differentiable and homogeneous of degree one, so that the production technology has constant returns to scale. With price-taking firms, this has the useful implication that "market structure" does not matter: Any number of firms may be operating with the same technology, or the firms may have heterogeneous technologies provided that each satisfies constant returns to scale. In both cases, we can describe the aggregate production technology for the economy using a transformation function F.

Given the transformation function F and producer prices p, the representative competitive firm maximizes profits over production vectors y:

$$\max_{y \in \mathbb{R}^n} p \cdot y \quad \text{subject to} \quad F(y) \leq 0. \tag{1.3}$$

Example 1.2. Consider an economy with two produced goods $\{1, 2\}$ and one labor good. We naively describe the production technology for the economy using production functions:

$$\tilde{y}_1 = A_1 \tilde{y}_2^{\alpha} \tilde{L}_1^{1-\alpha} \quad \text{and} \quad \tilde{y}_2 = A_2 \tilde{L}_2. \tag{1.4}$$

That is, labor is used to produce goods 1 and 2, and good 2 is additionally used to produce good 1. What is the induced transformation function $F(y_1, y_2, -L)$?[1] We can find a candidate by attempting to maximize the net output of one of the goods (say y_1) while respecting the constraints imposed by the production functions described above:

$$F(y_1, y_2, -L) \quad := \quad y_1 - \max_{\tilde{L}_1, \tilde{L}_2, \tilde{y}_2 \geq 0} A_1 \tilde{y}_2^{\alpha} \tilde{L}_1^{1-\alpha} \tag{1.5}$$

$$\text{subject to} \tag{1.6}$$

$$y_2 = A_2 \tilde{L}_2 - \tilde{y}_2 \tag{1.7}$$

$$L = \tilde{L}_1 + \tilde{L}_2. \tag{1.8}$$

[1] But note that F is not uniquely defined: Given any F, $\varphi \circ F$ for any strictly increasing function φ with $\varphi(0) = 0$ is a transformation function that describes the same production technology.

To interpret, the maximization problem requires that we maximize the net output of good 1, subject to producing net outputs $(y_2, -L)$ of the remaining goods.[2] $F(y_1, y_2, -L)$ is then the difference between the prescribed net output y_1 and the maximal net output of good 1. It is easy to verify that F is homogeneous of degree one and that $F(y_1, y_2, -L) \leq 0$ characterizes the set of production vectors that are feasible given the production functions (1.4) and free disposal.

Government. The government uses commodity taxes and a lump-sum tax to finance a vector of public spending $g \in \mathbb{R}^n$ and potentially correct market failures due to externalities. The government's implied budget constraint is

$$(q - p) \cdot x + T = p \cdot g. \tag{1.9}$$

Equilibrium. An equilibrium is this economy is essentially a Walrasian competitive equilibrium with taxes: a tuple (x, q, p, T) such that

(i) the government's budget constraint is satisfied;

(ii) the representative consumer chooses consumption vector x given q, T, and $\bar{x} = x$;

(iii) the representative firm chooses production vector y given p; and

(iv) all markets clear, $x + g = y$.

Pigouvian Tax Formula. The key optimality condition for relative prices in this environment is

$$\frac{p_i/q_i}{p_j/q_j} = \frac{1 + \frac{u_{\bar{x}_i}(x_*, x_*)}{u_{x_i}(x_*, x_*)}}{1 + \frac{u_{\bar{x}_j}(x_*, x_*)}{u_{x_j}(x_*, x_*)}}. \tag{1.10}$$

This formula indicates that the marginal rate of substitution between i and j, q_i/q_j, should be distorted downward from the marginal technical rate of substitution p_i/p_j whenever the normalized marginal externality from good i, $u_{\bar{x}_i}/u_{x_i}$, is large relative to the normalized marginal externality from good j, $u_{\bar{x}_j}/u_{x_j}$.

[2]The similarity with a standard Pareto optimality problem is no accident! Whether we are interested in characterizing the production possibilities frontier (as we are here) or the Pareto frontier in a given economy, the goal is to optimize one component of a vector subject to constraints on the remaining components.

2 Examples

Example 2.1. Suppose the economy has two goods, consumption c and labor l, and suppose that the representative consumer has preferences given by the utility function

$$v(c,l,\bar{c}) = c - \frac{l^{1+\eta}}{1+\eta} - \alpha\bar{c}. \tag{2.1}$$

Here aggregate consumption $\bar{c}$ imposes an additively separable externality on the representative consumer. The production technology is linear: $F(c,-l) = c - Al$.

To solve for the optimal corrective taxes, we begin by solving the competitive equilibrium given policy variables (q,p,T). The consumer's problem is

$$\max_{c,l\geq 0} c - \frac{l^{1+\eta}}{1+\eta} - \alpha\bar{c} \quad \text{subject to} \quad q_c c - q_l l + T \leq 0. \tag{2.2}$$

The first-order conditions imply

$$l = \left(\frac{q_l}{q_c}\right)^{\frac{1}{\eta}} \quad \text{and} \quad c = \left(\frac{q_l}{q_c}\right)^{\frac{1+\eta}{\eta}} - \frac{T}{q_c}. \tag{2.3}$$

Note that since the externality is additively separable in the consumer's utility function, it has no direct impact on the consumer's equilibrium consumption or labor supply decisions. The representative firm's problem is

$$\max_{c,l} p_c c - p_l l \quad \text{subject to} \quad c \leq Al. \tag{2.4}$$

The first-order condition implies $p_c = p_l/A$. Finally, recall the government's budget constraint:

$$(q_c - p_c)c - (q_l - p_l)l + T = p_c c^G + p_l l^G. \tag{2.5}$$

This equation can be solved simultaneously with the consumer's consumption choice to express the tax T as a function of prices (q,p) and exogenous government consumption $\left(c^G, l^G\right)$.

To determine optimal values for the policy variables (q,p,T), we solve the government's welfare maximization problem. In particular, the government optimizes over consumption c and labor l, internalizing the consumption externality and subject to the production technology:

$$\max_{c,l} (1-\alpha)c - \frac{l^{1+\eta}}{1+\eta} \quad \text{subject to} \quad c + c^g \leq A(l - l^g). \tag{2.6}$$

The first-order conditions imply

$$l^* = [(1-\alpha)A]^{\frac{1}{\eta}} \quad \text{and} \quad c^* = A\Big([(1-\alpha)A]^{\frac{1}{\eta}} - l^g\Big) - c^g. \tag{2.7}$$

To implement this allocation in equilibrium, we must set

$$\frac{q_l}{q_c} = (1-\alpha)A \quad \text{and} \quad \frac{p_l}{p_c} = A, \tag{2.8}$$

with the lump-sum tax T chosen to satisfy the government's budget constraint given allocation (c^*, l^*), chosen prices (q, c), and exogenous government consumption $\left(c^G, l^G\right)$. Note in particular that we arrive at one relation between relative prices that determines the optimal tax distortion:

$$\frac{p_c/q_c}{p_l/q_l} = 1-\alpha = \frac{1 + \frac{v_{\bar{c}}(c^*,l^*,c^*)}{v_c(c^*,l^*,c^*)}}{1 + \frac{v_{\bar{l}}(c^*,l^*,c^*)}{v_l(c^*,l^*,c^*)}}. \tag{2.9}$$

The last equality emphasizes the connection with the general Pigouvian tax formula.[3] We can implement this relation (and the first-best allocation) using a variety of different tax instruments. For example, taxing consumption suffices:

$$q_c = (1+\tau_c)\,p_c \quad \text{and} \quad q_l = p_l, \tag{2.10}$$

where $1+\tau_c = 1/(1-\alpha)$. This tax increases the consumer's price of consumption and distorts downward equilibrium consumption, aligning the private and social marginal benefits of consumption. These two equations, along with the producer optimality condition $p_c = p_l/A$ and a standard Walrasian price normalization, determine equilibrium consumer and producer prices. The lump-sum tax T can then be computed from the government's budget constraint. Alternatively, we can tax labor supply:

$$q_c = p_c \quad \text{and} \quad q_l = (1+\tau_l)\,p_l, \tag{2.11}$$

were $1+\tau_l = 1-\alpha$. Again, this tax has the effect of distorting downward equilibrium consumption.

[3] And note in this case how much easier it is to derive the optimal price distortions from the general formula than by solving for the equilibrium and the social optimum! This only holds because there are no externalities associated with labor, and the marginal utility of consumption and the marginal consumption externality are both constant.

Example 2.2. In this example, we consider conditions under which the general equilibrium analysis above can be reduced to the partial equilibrium graphical analysis often seen in undergraduate classes. We begin with the general model, and we suppose that the representative consumer's utility function has the additively separable, quasilinear representation

$$u(x, \bar{x}) = x_1 + \sum_{i=2}^{n} u^i(x_i, \bar{x}_i). \tag{2.12}$$

The first-order conditions to the consumer's problem are then

$$x_1 = \sum_{i=2}^{n} \frac{q_i}{q_1} x_i - \frac{T}{q_1}, \tag{2.13}$$

$$u^i_{x_i}(x_i, \bar{x}_i) = q_i \quad i \in \{2, \ldots, n\}. \tag{2.14}$$

Now $\bar{x}_i = x_i$ in equilibrium, so we find that the equilibrium inverse demand function $x_i \mapsto u^i_{x_i}(x_i, x_i)$ is independent of the quantities in markets $j \neq i$ (because of additive separability) and the consumer's wealth (because of quasilinearity). Similarly, the "social marginal benefit" function $x_i \mapsto u^i_{x_i}(x_i, x_i) + u^i_{\bar{x}_i}(x_i, x_i)$ is also independent of the quantities in markets $j \neq i$. Suppose also that the aggregate transformation function takes the separable form

$$F(y) = \tag{2.15}$$