Recitation 3: Diamond-Mirrlees II + Pigou

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Recitation Plan: Integrate externalities into the Diamond-Mirrlees II optimal tax formulas and derive the Sandmo (1975) additivity result

1 General Model

Consumption. The economy has a continuum of heterogeneous consumers, where each consumer has a type h that belongs to a finite set H. Let π^h denote the proportion of consumers of type h. Each consumer has preferences defined over her own net consumption vector $x^h \in X$ and the average net consumption vector in the population $\bar{x} \in X$, where $X \subseteq \mathbb{R}^n$ is assumed convex with a non-empty interior. These preferences are represented by a utility function $u^h\left(x^h, \bar{x}\right)$, assumed differentiable in all arguments and concave and locally non-satiated in x^h for any $\bar{x}$. Since $\bar{x}$ directly enters u, we have the possibility of consumption externalities – one consumer's choice can directly affect another's utility.

Given consumer prices $q \in \mathbb{R}^n$, a lump-sum tax $T^h \in \mathbb{R}$, and the average consumption choice $\bar{x}$, a type h consumer solves

$$\max_{x^h \in X} u^h\left(x^h, \bar{x}\right) \quad \text{subject to} \quad q \cdot x^h + T^h \leq 0.$$

Throughout, we assume that the solution occurs at an interior point of X, so the budget constraint is the only active constraint.

Production. The production technology is described by the transformation function $F : \mathbb{R}^n \to \mathbb{R}$, where a net output vector is feasible if and only if $F(y) \leq 0$. We assume that F is differentiable and homogeneous of degree one, so that the production technology has constant returns to scale.

Given the transformation function F and producer prices p, the representative competitive firm maximizes profits over production vectors y:

$$\max_{y \in \mathbb{R}^n} p \cdot y \quad \text{subject to} \quad F(y) \leq 0.$$

Government. The government uses commodity taxes and a uniform lump-sum tax to finance a vector of public spending $g \in \mathbb{R}^n$ and potentially correct market failures due to externalities. The government's implied budget constraint is

$$(q - p) \cdot \bar{x} + T = p \cdot g.$$

Equilibrium. An equilibrium is this economy is essentially a Walrasian competitive equilibrium with taxes: a tuple $\left(\left(x^h\right)_{h \in H}, z, q, p, T\right)$ such that

(i) the government's budget constraint is satisfied;

(ii) each type h consumer chooses consumption vector x^h given q and T;

(iii) the representative firm chooses production vector y given p; and

(iv) all markets clear, $\sum_h x^h \pi^h + g = y$.

2 Optimal Tax Formulas

Below we derive optimal tax formulas generalizing those of Diamond and Mirrlees (1971), and we use them to study how externalities affect the structure of optimal commodity taxes. Following the dual approach, let $V^h(q, I, \bar{x})$ denote the indirect utility function for type h consumers, and let $x^h(q, I, \bar{x})$ and $\hat{x}^h(q, u, \bar{x})$ denote the Marshallian and Hicksian demand functions. Making use of arguments from previous recitations to justify government control of production, we can write the government's second-best Pareto problem as follows:

$$\begin{aligned} \max_{q,I} \mathbb{E}_h \left[V^h(q, I, \bar{x}) \lambda^h\right] \quad \text{subject to} \quad & F(\bar{x} + g) \leq 0, \\ & \bar{x} = \mathbb{E}_h \left[x^h(q, I, \bar{x})\right]. \end{aligned}$$

Here $\mathbb{E}_h$ denotes an expectation with respect to the distribution of types. This problem differs from the standard commodity taxation problem only by the inclusion of $\bar{x}$ in the indirect utility and demand functions. But note that $\bar{x}$ must satisfy its own fixed-point equation, so to derive correct optimal tax formulas we have to be careful in how we deal with this additional condition. One option would be to include $\bar{x}$ as a choice variable for the government and keep the fixed-point equation for $\bar{x}$ as a constraint. I follow a different strategy: Keeping with the spirit of the dual approach, I appeal to the Implicit Function Theorem and let $\bar{x}(q, I)$ denote the solution to the fixed-point equation, which under appropriate conditions is unique and continuously differentiable in a neighborhood of the solution to the government's problem. The government's problem can then be written

$$\max_{q,I} \sum_{h \in H} V^h(q, I, \bar{x}(q, I)) \lambda^h \pi^h \quad \text{subject to} \quad F(\bar{x}(q, I) + g) \leq 0.$$

Letting $\kappa > 0$ denote the multiplier on the feasibility constraint, the first-order condition with respect to q_i is

$$0 = \mathbb{E}_h \left[\lambda^h \frac{\partial V^h}{\partial q_i} + \lambda^h \frac{\partial V^h}{\partial \bar{x}} \frac{d\bar{x}}{dq_i} \right] - \kappa \sum_{j=1}^{n} \frac{\partial F}{\partial y_j} \mathbb{E}_h \left[\frac{\partial x_j^h}{\partial q_i} + \frac{\partial x_j^h}{\partial \bar{x}} \frac{d\bar{x}}{dq_i} \right].$$

Using Roy's Identity and the Slutsky Equation on the first and second terms, respectively, gives

$$0 = \frac{1}{\kappa} \mathbb{E}_h \left[-\lambda^h x_i^h \frac{\partial V^h}{\partial I} + \lambda^h \frac{\partial V^h}{\partial \bar{x}} \frac{d\bar{x}}{dq_i} \right] - \sum_{j=1}^{n} \frac{\partial F}{\partial y_j} \mathbb{E}_h \left[\frac{\partial \hat{x}_j^h}{\partial q_i} - x_i^h \frac{\partial x_j^h}{\partial I} + \frac{\partial x_j^h}{\partial \bar{x}} \frac{d\bar{x}}{dq_i} \right].$$

Substituting the producer price $p_j = \partial F / \partial y_j$ and using Slutsky symmetry and the homogeneity of degree zero identity $\sum_j q_j \left(\partial \hat{x}_i^h / \partial q_j \right) = 0$ on the second term, we can rearrange:

$$\sum_{j=1}^{n} t_j \mathbb{E}_h \left[\frac{\partial \hat{x}_j^h}{\partial q_i} \right] = \bar{x}_i \mathbb{E}_h \left[\frac{x_i^h}{\bar{x}_i} \left(\frac{\lambda^h}{\kappa} \frac{\partial V^h}{\partial I} - \sum_{j=1}^{n} p_j \frac{\partial x_j^h}{\partial I} \right) \right]$$
$$- \mathbb{E}_h \left[\frac{\lambda^h}{\kappa} \frac{\partial V^h}{\partial \bar{x}} \frac{d\bar{x}}{dq_i} - \sum_{j=1}^{n} p_j \frac{\partial x_j^h}{\partial \bar{x}} \frac{d\bar{x}}{dq_i} \right]$$

For one final manipulation, use the budget constraint identities $\sum_j q_j \left(\partial x_j^h / \partial I \right) = 1$ and $\sum_j q_j \left(\partial x_j^h / \partial \bar{x} \right) = 0$ to write

$$\sum_{j=1}^{n} t_j \mathbb{E}_h \left[\frac{\partial \hat{x}_j^h}{\partial q_i} \right] = \bar{x}_i \mathbb{E}_h \left[\frac{x_i^h}{\bar{x}_i} \left(\frac{\lambda^h}{\kappa} \frac{\partial V^h}{\partial I} - 1 + \sum_{j=1}^{n} t_j \frac{\partial x_j^h}{\partial I} \right) \right] \tag{DM}$$

$$-\mathbb{E}_h\left[\frac{\lambda^h}{\kappa}\frac{\partial V^h}{\partial \bar{x}}\frac{d\bar{x}}{dq_i}+\sum_{j=1}^{n} t_j \frac{\partial x_j^h}{\partial \bar{x}}\frac{d\bar{x}}{dq_i}\right].$$

This is precisely the optimal tax formula from Diamond and Mirrlees (1971), but with an additional term on the right side that accounts for the effect of the externalities on the optimal tax structure. For intuition (which also applies in the case with no externalities!), note that we could heuristically derive this equation by solving the following version of the government's problem:

$$\max_{q,I}\mathbb{E}_h\left[V^h\left(q,I,\bar{x}\left(q,I\right)\right)\lambda^h\right] \quad \text{subject to} \quad (q-p)\cdot\bar{x}-I=p\cdot g.$$

Here we maximize Pareto-weighted welfare over consumer prices and lump sum income, but subject to the government's budget constraint and *holding producer prices fixed*. Letting κ denote the multiplier on the budget constraint, the first-order condition with respect to q_i is

$$-\mathbb{E}_h\left[\frac{\lambda^h}{\kappa}\left(-x_i^h\frac{\partial V^h}{\partial I}+\frac{\partial V^h}{\partial \bar{x}}\frac{d\bar{x}}{dq_i}\right)\right]=\bar{x}_i+\sum_{j=1}^{n} t_j\mathbb{E}_h\left[\frac{\partial x_j^h}{\partial q_i}+\frac{\partial x_j^h}{\partial \bar{x}}\frac{d\bar{x}}{dq_i}\right]. \tag{DM'}$$

Note that the left side captures the direct effect of the price increase on indirect utilities, while the right side captures the effect on government revenues. This equation can be rearranged to give the Diamond-Mirrlees formula (DM), so the latter captures exactly the "direct effects and fiscal externalities" intuition discussed in class.

Continuing with formula (DM'), we can get a sense for how externalities affect optimal commodity taxes by considering a special case. Suppose the utility functions take the weakly separable form $u^h\left(x^h,\bar{x}\right)=\tilde{u}^h\left(w^h\left(x^h\right),\bar{x}\right)$. Then the Marshallian demand functions $x^h\left(q,I\right)$ do not depend on average consumption $\bar{x}$, and average consumption is no longer defined by a fixed-point equation:

$$\bar{x}\left(q,I\right)=\sum_{h\in H}x^h\left(q,I\right)\pi^h.$$

Applying these observations, the formula (DM') becomes

$$\mathbb{E}_h\left[\frac{\lambda^h}{\kappa}x_i^h\frac{\partial V^h}{\partial I}\right]-\sum_{j=1}^{n}\mathbb{E}_h\left[\frac{\lambda^h}{\kappa}\frac{\partial V^h}{\partial \bar{x}_j}\right]\mathbb{E}_h\left[\frac{\partial x_j^h}{\partial q_i}\right]=\bar{x}_i+\sum_{j=1}^{n} t_j\mathbb{E}_h\left[\frac{\partial x_j^h}{\partial q_i}\right]. \tag{DM''}$$

Define $\tilde{t}_j := t_j + \mathbb{E}_h\left[\frac{\lambda^h}{\kappa}\frac{\partial V^h}{\partial \bar{x}_j}\right]$. Then formula (DM') can be written

$$\sum_{j=1}^{n} \tilde{t}_j \mathbb{E}_h\left[\frac{\partial \hat{x}_j^h}{\partial q_i}\right] = \mathbb{E}_h\left[x_i^h \frac{\lambda^h}{\kappa}\frac{\partial V^h}{\partial I}\right] - \bar{x}_i.$$

Structurally, this equation is *exactly* the standard Diamond-Mirrlees formula for economies without externalities. This result was first derived by Sandmo (1975), and it implies that the optimal taxes take additive form $t_j = \tilde{t}_j + t_j^P$, where the "Pigouvian correction" is $t_j^P := -\mathbb{E}_h\left[\frac{\lambda^h}{\kappa}\frac{\partial V^h}{\partial \bar{x}_j}\right]$.

Two observations: First, the additive property demonstrates that we should *not* tax complements or subsidize substitutes for a good with a negative externality. We saw this when we considered Pigouvian taxation with type-specific lump-sum taxes, and it is notable that it continues to hold with fewer tax instruments. Second, the Pigouvian correction depends directly on the Pareto weights λ^h. Corrective taxes are now directly sensitive to the government's redistributive objective, contrary to what we found with type-specific lump-sum taxes. Intuitively, the government can no longer handle redistribution "behind the scenes" with lump-sum taxes – it compensates by placing larger corrective taxes on goods whose negative consumption externalities are borne by types with higher Pareto weights.

References

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