# Recitation 4: Chamley-Judd Revisited

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**Recitation Plan:** Review the zero capital taxation result of Chamley (1986) and the criticism of Straub and Werning (2020)

## 1 Model

**Consumption.** The economy exists in discrete time $t \in \{0, 1, \dots\}$ and consists of a representative agent. The agent has intratemporal preferences over consumption $c_t$ and labor supply $n_t$ represented by a utility function $U(c_t, n_t)$. We assume that $U$ is such that consumption and leisure are both normal goods:

$$\frac{U_{cc}}{U_c} - \frac{U_{nc}}{U_n}, \frac{U_{cn}}{U_c} - \frac{U_{nn}}{U_n} \leq 0.$$

The agent's intertemporal utility satisfies the stationary recursion

$$V_t = W(U_t, V_{t+1}) \quad \text{and} \quad V_t = \mathcal{V}\left((U_s)_{s=t}^{\infty}\right).$$

Finally, it is helpful to define the "discount factor" applied to $t+1$ continuation utility $V_{t+1}$ in period $t$ when the agent is in a "steady state" with constant continuation utilities:

$$\bar{\beta}(V) := W_V\left(\bar{U}(V), V\right), \quad \text{where} \quad V = W\left(\bar{U}(V), V\right).$$

At each date $t$, the agent can consume $c_t$, supply labor $n_t$, save capital $k_{t+1}$, and purchase government bonds $b_{t+1}$, taking the post-tax wage $w_t$ and the post-tax return on capital and bonds $R_{t+1}$ as given.[1] The agent then solves

$$\max_{(c_t, n_t, k_{t+1}, b_{t+1})_{t=0}^{\infty}} \mathcal{V}\left((U_s)_{s=0}^{\infty}\right) \quad \text{subject to} \quad c_0 + k_1 + b_1 \leq w_0 n_0 + R_0 k_0 + R_0^b b_0,$$

$$c_t + k_{t+1} + b_{t+1} \leq w_t n_t + R_t(k_t + b_t) \quad t \geq 1,$$

[1] By a standard arbitrage argument, the post-tax returns on capital and bonds must be equal in every period after the initial period.

$$\lim_{t\to\infty}(k_{t+1}+b_{t+1})\prod_{s=1}^{t}R_t^{-1}=0.$$

Note that the $t=0$ budget constraint must be written separately: Initial asset holdings $k_0$ and $b_0$ are fixed, so we do not have an arbitrage argument that requires the post-tax return on capital $R_0$ equal the post-tax return on bonds $R_0^b$.

**Production.** The production technology is described by the production function $F(k_t, n_t)$ for the final consumption good, assumed differentiable and homogeneous of degree one. Production is competitive, so the pre-tax wage $w_t^*$ and return on capital $R_t^*$ are determined by marginal products:

$$w_t^* = F_n(k_t, n_t) \quad \text{and} \quad R_t^* = F_k(k_t, n_t).$$

The pre- and post-tax wage and returns are related by the identities

$$w_t = \left(1-\tau_t^n\right)w_t^* \quad \text{and} \quad R_t = (1-\tau_t)\left(R_t^*-1\right)+1. \tag{1}$$

Here $\tau_t^n$ is the tax on labor income, while $\tau_t$ is the tax on (net) asset returns.

**Government.** The government uses taxes $\left(\tau_t^n, \tau_t\right)_{t=0}^{\infty}$ and $R_0^b$ to finance a stream of public spending $(g_t)_{t=0}^{\infty}$. The government's implied budget constraints are

$$\begin{aligned} g_0 + R_0^b b_0 &\le \tau_0^n w_0^* + \tau_0\left(R_0^*-1\right)k_0 + b_1, \\ g_t + R_t b_t &\le \tau_t^n w_t^* + \tau_t\left(R_t^*-1\right)k_t + b_{t+1} \quad t \ge 1. \end{aligned}$$

Following Chamley (1986), the government is also constrained by a lower bound on the after-tax rate of return: $R_t \ge (1-\bar{\tau})\left(R_t^*-1\right)+1$ for $t \ge 1$. The implied bound on the tax on net asset returns is $\tau_t \le \bar{\tau}$.

**Equilibrium.** An equilibrium is a tuple $\left(\left(c_t, n_t, k_{t+1}, b_{t+1}, w_t^*, R_{t+1}^*, \tau_t^n, \tau_t\right)_{t=0}^{\infty}, R_0^*, R_0^b\right)$ such that

(i) the government's budget constraint is satisfied;

(ii) pre-tax prices are determined by marginal products;

(iii) the agent chooses $(c_t, n_t, k_{t+1}, b_{t+1})_{t=0}^{\infty}$ given post-tax prices; and

(iv) the resource constraint is satisfied in each period, $c_t + g_t + k_{t+1} \le F(k_t, n_t)$.

## 2 Capital Taxation in the Long Run

To characterize the optimal path of taxes $\left(\tau_t^n, \tau_t\right)_{t=0}^{\infty}$, we follow the primal approach: By a standard argument, an allocation $(c_t, n_t, k_{t+1})_{t=0}^{\infty}$ can be implemented in a competitive equilibrium with taxes if and only if

$$\begin{aligned} c_t + g_t + k_{t+1} &\leq F(k_t, n_t) + (1-\delta)k_t && t \geq 0, \\ R_0 &\geq (1-\bar{\tau})(F_k(k_0, n_0) - 1) + 1, \\ \frac{\mathcal{V}_{ct}}{\mathcal{V}_{c(t+1)}} &\geq (1-\bar{\tau})(F_k(k_{t+1}, n_{t+1}) - 1) + 1 && t \geq 0, \\ \sum_{t=0}^{\infty} (\mathcal{V}_{ct} c_t + \mathcal{V}_{nt} n_t) &= \mathcal{V}_{c0}\left(R_0 k_0 + R_0^b b_0\right). \end{aligned}$$

The first set of constraints ensures feasibility, the second set of constraints ensures that the capital tax bound $R_t \geq 1$ is satisfied, and the final constraint ensures implementability. The government's problem is then to choose the allocation $(c_t, n_t, k_{t+1})_{t=0}^{\infty}$ to maximize $t = 0$ utility $V_0$ subject to the feasibility constraints, capital tax constraints, and the implementability constraint.

We begin by recalling Chamley's (1986) result:

**Theorem** (Chamley, 1986, Theorem 1)**.** Let $\tilde{\Lambda}_t$ denote the Lagrange multiplier on the period $t$ resource constraint, and let $\Lambda_t := \tilde{\Lambda}_t / \mathcal{V}_{ct}$. Suppose $c_t, k_{t+1} > 0$ for $t \geq 0$, and suppose that for $t > T$ the capital tax constraints are non-binding. Then if $\Lambda_t \to \Lambda > 0$, $R_t / R_t^* \to 1$.

*Proof.* With $t > T$, the first-order condition for $t+1$ capital $k_{t+1}$ is

$$\tilde{\Lambda}_t = \tilde{\Lambda}_{t+1} R_{t+1}^* \iff \mathcal{V}_{ct} \Lambda_t = \mathcal{V}_{c(t+1)} \Lambda_{t+1} R_{t+1}^*.$$

The post-tax return $R_{t+1}$ is defined by the agent's Euler equation:

$$\mathcal{V}_{ct} = \mathcal{V}_{c(t+1)} R_{t+1}.$$

Dividing these equations yields $R_{t+1}^* / R_{t+1} = \Lambda_t / \Lambda_{t+1} \to 1$. ∎

The theorem states that if the capital tax constraint is asymptotically non-binding and the government's (normalized) marginal value of resources in period $t$ converges to a positive constant, then the optimal capital tax $\tau_t$ must converge to zero.[2] Straub and Werning (2020)

[2]Note that if $\bar{\tau} = 1$, then the condition that the capital tax constraint is asymptotically non-binding can be

offer two key criticisms of this result: First, in the standard case in which intratemporal utility $U$ is isoelastic and additively separable and intertemporal aggregation $W$ is additively separable, the multiplier $\Lambda_t$ need not converge to a positive value and/or the capital tax constraint may not be asymptotically non-binding. As a result, the optimal capital tax may satisfy $\tau_t = \bar{\tau}$ in all periods. The proof of this result is lengthy, and I refer you to the appendix of Straub and Werning (2020). Second, Straub and Werning (2020) show that even when Chamley's (1986) result applies, as long as intertemporal aggregation $W$ is not additively separable, the zero long-run capital tax is also accompanied by zero long-run wealth or zero long-run labor taxation:

**Theorem** (Straub and Werning, 2020, Proposition 6)**.** Suppose the optimal allocation converges to an interior steady state, and suppose that for $t > T$ the capital tax constraints are non-binding. Then $\tau_t \to 0$, and if $\bar{\beta}'(V) \neq 0$ at the steady-state continuation utility $V$, then either

(i) private wealth converges to zero, $a_t := k_t + b_r \to 0$; or

(ii) the allocation converges to the first-best, with $\tau^n_t \to 0$.

*Proof.* The proof follows from an examination of the first-order conditions to the government's problem. First, we define notation from the agent's preferences: Given the optimal allocation $(c_t, n_t, k_{t+1})_{t=0}^{\infty}$, define the period-$t$ discount rate $\beta_t := \prod_{s=0}^{t-1} W_V(U_s, V_{s+1})$. Then using the intertemporal recursion $\mathcal{V}_{ct} = \beta_t W_{Ut} U_{ct}$ and $\mathcal{V}_{nt} = \beta_t W_{Ut} U_{nt}$, the implementability constraint can be written in the more familiar form

$$\sum_{t=0}^{\infty} \beta_t W_{Ut} (U_{ct} c_t + U_{nt} n_t) = W_{U0} U_{c0} \left(R_0 k_0 + R_0^b b_0\right).$$

The government's problem can then be stated

$$\max_{(V_t, c_t, n_t, k_{t+1})_{t=0}^{\infty}, R_0, R_0^b} V_0 \tag{2}$$

$$\text{subject to} \tag{3}$$

$$V_t = W\left(U\left(c_t, n_t\right), V_{t+1}\right) \qquad t \geq 0, \tag{4}$$

$$c_t + g_t + k_{t+1} \leq F\left(k_t, n_t\right) \qquad t \geq 0, \tag{5}$$

$$\sum_{t=0}^{\infty} \beta_t W_{Ut} (U_{ct} c_t + U_{nt} n_t) = W_{U0} U_{c0} \left(R_0 k_0 + R_0^b b_0\right), \tag{6}$$

removed.

$$\tau_t \leq \bar{\tau} \qquad t \geq 0. \tag{7}$$

Let $\beta_t \nu_t$ be the multiplier on the period-$t$ recursion constraint, let $\beta_t \lambda_t$ be the multiplier on the period-$t$ resource constraint, and let $\mu$ be the multiplier on the implementability constraint. Then for $t > T$, the first-order conditions for $V_{t+1}$, $c_t$, $n_t$, and $k_{t+1}$ are

$$\begin{aligned}
(V_{t+1}) \quad & 0 = -\nu_t + \nu_{t+1} - \mu A_{t+1}, \\
(c_t) \quad & 0 = \nu_t W_{Ut} U_{ct} - \mu W_{Ut} \left( U_{ct} + U_{cct} c_t + U_{nct} n_t \right) - \mu B_t U_{ct} - \lambda_t, \\
(n_t) \quad & 0 = \nu_t W_{Ut} U_{nt} - \mu W_{Ut} \left( U_{nt} + U_{cnt} c_t + U_{nnt} n_t \right) - \mu B_t U_{nt} + \lambda_t F_{nt}, \\
(k_{t+1}) \quad & 0 = -\lambda_t + \lambda_{t+1} W_{Vt} F_{k(t+1)},
\end{aligned}$$

where we have used the assumption that the capital tax constraints are non-binding and

$$\begin{aligned}
A_{t+1} &:= \frac{1}{\beta_{t+1}} \frac{\partial}{\partial V_{t+1}} \sum_{s=0}^{\infty} \beta_s W_{Us} \left( U_{cs} c_s + U_{ns} n_s \right) \\
B_t &:= \frac{1}{\beta_t} \sum_{s=0}^{\infty} \frac{\partial \left( \beta_s W_{U_s} \right)}{\partial U_t} \left( U_{cs} c_s + U_{ns} n_s \right).
\end{aligned}$$

Now we suppose that $(c_t, n_t, k_{t+1}) \to (c, n, k)$, which implies that utilities, continuation values, and their derivatives also converge. Assets $a_t := k_t + b_t$ converge, and the limit can be found by using a period $t+1$ version of the implementability constraint:

$$a_{t+1} = \frac{\sum_{s=t+1}^{\infty} \beta_s W_{U_s} \left( U_{cs} c_s + U_{ns} n_s \right)}{W_{U(t+1)} U_{c(t+1)} \beta_{t+1} R_{t+1}} \to \frac{U_c c + U_n n}{\left( 1 - \bar{\beta}(V) \right) U_c R} =: a,$$

where the limit holds because the sum is asympotically a geometric series. A similar argument implies $A_{t+1} \to A$ and $B_t \to B$, where

$$A = \frac{\bar{\beta}'(V)}{\bar{\beta}(V)} W_U U_c R a.$$

Taking the limits for all allocation variables in the first-order conditions, we have

$$\begin{aligned}
(V) \quad & 0 = -\nu_t + \nu_{t+1} - \mu A, \\
(c) \quad & 0 = \nu_t - \mu \left( 1 + \frac{U_{cc} c}{U_c} + \frac{U_{nc} n}{U_c} \right) - \mu \frac{B}{W_U} U_c - \frac{\lambda_t}{W_U U_c}, \\
(n) \quad & 0 = \nu_t - \mu \left( 1 + \frac{U_{cn} c}{U_n} + \frac{U_{nn} n}{U_n} \right) - \mu \frac{B}{W_U} + \lambda_t \frac{F_n}{W_U U_n},
\end{aligned}$$

$$(k) \quad 0 = -\lambda_t + \lambda_{t+1}\bar{\beta}(V) F_k.$$

Making use of the conditions for $V$, $c$, and $k$, we find

$$\bar{\beta}(V) F_k - 1 = \frac{\lambda_t}{\lambda_{t+1}} - 1 = -\frac{W_U U_c}{\lambda_{t+1}} \mu A.$$

To prove the theorem, we first show that capital taxes are indeed zero, $\bar{\beta}\left(\bar{V}\right) F_k = 1$. If $A = 0$ or $\mu = 0$, then this is immediate from the equation above. Otherwise, the $V$ condition requires that $\nu_t \to \pm\infty$, and hence that $\lambda_t \to \pm\infty$. We again recover $\bar{\beta}\left(\bar{V}\right) F_k = 1$ using the equation above.

We can complete the argument by showing that $a \neq 0$ and $\bar{\beta}'(V) \neq 0$ imply $\tau_t^n = 0$. Using the conditions for $c$ and $n$, the labor tax satisfies

$$\lambda_t \tau^n = \mu \frac{W_U U_n}{F_n} \left[ \frac{U_{cc} c}{U_c} + \frac{U_{nc} n}{U_c} - \left( \frac{U_{cn} c}{U_n} + \frac{U_{nn} n}{U_n} \right) \right].$$

To see that $\tau^n = 0$, note that $\mu = 0$ implies that the economy is first-best, which immediately implies $\tau_t^n = 0 \ \forall t \geq 0$. Suppose instead that $\mu \neq 0$. If $\lambda_t \to \pm\infty$, then the equation above immediately implies $\tau^n = 0$. Suppose instead that $\lambda_t \to \lambda \in \mathbb{R}$.[3] This implies $\nu_t \to \nu$, and hence that $A = 0$. But this is a contradiction since we assume that $a, \bar{\beta}'(V) \neq 0$. ■

This result suggests caution in interpreting Chamley's (1986) zero capital taxation result away from the "knife-edge" case of additively separable intertemporal aggregation: Though zero capital taxation may be optimal in the long run, this long run must also feature either zero labor taxation (symmetric treatment of capital and labor) or zero private wealth!

## References


Chamley, C. (1986). Optimal taxation of capital income in general equilibrium with infinite lives. *Econometrica: Journal of the Econometric Society*, 607–622.

Straub, L., & Werning, I. (2020). Positive long-run capital taxation: Chamley-judd revisited. *American Economic Review*, *110*(1), 86–119.


[3]This is the only other possibility, because zero capital taxation implies $\frac{\lambda_t}{\lambda_{t+1}} \to 1$.