Recitation 5: Nonlinear Taxation I

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Recitation Plan: Mathematically formulate the nonlinear income taxation problem with two types, and discuss the generalization of Stiglitz (1982) with endogenous wages

1 Model

Consumption. The economy has a measure μ_i of agents of type $i \in \{1,2\}$, where $\mu_1 + \mu_2 = 1$. Each agent of type i has preferences over consumption c_i and labor n_i given by the utility function $u^i(c_i, n_i)$, assumed twice continuously differentiable, strictly concave, strictly increasing in c_i, and strictly decreasing in n_i. Given an income tax schedule T and a pre-tax wage w_i, a type i agent solves

$$\max_{c_i, n_i} u^i(c_i, n_i) \quad \text{subject to} \quad c_i \leq w_i n_i - T(w_i n_i).$$

Production. The production technology is described by the production function $F(n_1, n_2)$ for the final consumption good, assumed twice continuously differentiable and homogeneous of degree one. Production is competitive, so the pre-tax wages (w_1, w_2) are determined by the marginal products:

$$w_i = F_{n_i}(\mu_1 n_1, \mu_2 n_2),$$

or equivalently

$$w_2 = f'(n) \quad \text{and} \quad w_1 = f(n) - n f'(n),$$

where $n := \mu_2 n_2 / \mu_1 n_1$ and $f(n) := F(1, n)$.

Government. The government uses the income tax schedule T to finance exogenous government expenditures g and to redistribute. The government's budget constraint is

$$g \leq \mu_1 T(w_1 n_1) + \mu_2 T(w_2 n_2).$$

Equilibrium. An equilibrium is a tuple $\left((c_i, n_i)_{i=1,2}, T\right)$ such that

(i) the government's budget constraint is satisfied;

(ii) pre-tax wages are determined by marginal products;

(iii) each type i agent chooses (c_i, n_i) given the pre-tax wage w_i and the tax schedule T; and

(iv) the resource constraint is satisfied in each period, $\mu_1 c_1 + \mu_2 c_2 + g \leq F(\mu_1 n_1, \mu_2 n_2)$.

2 Special Case: Linear Production

First consider the case in which the production function F is linear: $F(\mu_1 n_1, \mu_2 n_2) = w_1 \mu_1 n_1 + w_2 \mu_2 n_2$, where $w_2 > w_1$. Additionally assume the following single-crossing condition:

$$\mathrm{MRS}^2(c, y) := -\frac{1}{w_2} \frac{u_n^2(c, y/w_2)}{u_c^2(c, y/w_2)} < -\frac{1}{w_1} \frac{u_n^1(c, y/w_1)}{u_c^1(c, y/w_1)} =: \mathrm{MRS}^1(c, y) \quad \forall (c, y) \gg 0.$$

This condition implies that the indifference curves for a type 2 agent are flatter than those for a type 1 agent in (y, c)-space, and it will imply that type 2 agents will earn higher incomes under a Pareto efficient income tax.

The Pareto efficient income taxation problem is as follows: Choose the tax schedule T to maximize the equilibrium utility of type 2 agents, subject to the government's budget constraint and the constraint that type 1 agents achieve utility $\bar{u}^1$ in equilibrium. This is a difficult problem! As stated, the "choice variable" is an infinite-dimensional object (the tax schedule T). To simplify, we make use of a change of variables known in mechanism design as the Revelation Principle: Fix a tax schedule T, and let (c_i, n_i) denote the consumption bundle chosen by an agent of type i in equilibrium. Since agents optimize, these consumption bundles must satisfy the "incentive constraints"

$$u^i(c_i, n_i) \geq u^i\left(c_{i'}, \frac{w_{i'}}{w_i} n_{i'}\right) \quad i \neq i'.$$

In words, a type i agent must weakly prefer her own bundle to that of a type i' agent; otherwise, she would have chosen the type i' agent's bundle. Conversely, suppose that we find consumption bundles $(c_i, n_i)_{i=1,2}$ that satisfy the incentive constraints above as well as the resource constraint. By the single-crossing condition, we must have $w_2 n_2 > w_1 n_1$, and we can define an income tax schedule by

$$T(y) := \begin{cases} w_i n_i - c_i & \text{if } y = w_i n_i \text{ for } i = 1, 2, \\ \infty & \text{else.} \end{cases}$$

It is straightforward to verify that if the consumption bundles $(c_i, n_i)_{i=1,2}$ satisfy the incentive compatibility conditions, then an agent of type i will choose bundle (c_i, n_i) when confronted with the income tax schedule T. As a result, instead of formulating the Pareto efficiency problem as an optimization problem over tax schedules, we can instead choose an allocation $(c_i, n_i)_{i=1,2}$ subject to incentive compatibility constraints:

$$\begin{aligned} \max_{(c_i, n_i)_{i=1,2}} u^2(c_2, n_2) \quad \text{subject to} \quad & u^1(c_1, n_1) \geq \bar{u}^1, \\ & u^i(c_i, n_i) \geq u^i\left(c_{i'}, \frac{w_{i'}}{w_i} n_{i'}\right) \qquad i \in \{1, 2\}, i \neq i', \\ & \mu_1 c_1 + \mu_2 c_2 + g \leq F(\mu_1 n_1, \mu_2 n_2). \end{aligned}$$

We can characterize properties of the solution using first-order conditions. Let $\eta > 0$ denote the multiplier on the utility constraint, let $\lambda_i \geq 0$ denote the multiplier on type i's incentive constraint, and let $\gamma > 0$ denote the multiplier on the resource constraint. The first-order conditions are

$$\begin{aligned} (c_1) \quad & 0 = (\eta + \lambda_1) u_c^1(c_1, n_1) - \lambda_2 u_c^2\left(c_1, \frac{w_1}{w_2} n_1\right) - \gamma \mu_1, \\ (n_1) \quad & 0 = (\eta + \lambda_1) u_n^1(c_1, n_1) - \frac{w_1}{w_2} \lambda_2 u_n^2\left(c_1, \frac{w_1}{w_2} n_1\right) + \gamma w_1 \mu_1, \\ (c_2) \quad & 0 = (1 + \lambda_2) u_c^2(c_2, n_2) - \lambda_1 u_c^1\left(c_2, \frac{w_2}{w_1} n_2\right) - \gamma \mu_2, \\ (n_2) \quad & 0 = (1 + \lambda_2) u_n^2(c_2, n_2) - \frac{w_2}{w_1} \lambda_1 u_n^1\left(c_2, \frac{w_2}{w_1} n_2\right) + \gamma w_2 \mu_2, \end{aligned}$$

Using these conditions, we can show by contradiction that we cannot have $\lambda_1, \lambda_2 > 0$, i.e., at most one incentive constraint can bind. We will suppose that $\bar{u}^1$ is sufficiently high so that

$\lambda_2 > 0 = \lambda_1$: The government wishes to redistribute from high-wage type 2 to low-wage type 1 agents, so the type 2 agents' incentive constraint would not be satisfied in the first-best allocation. In this case, the type 2 first-order conditions are

$$
\begin{aligned}
(c_2) \quad 0 &= (1+\lambda_2)u_c^2(c_2, n_2) - \gamma\mu_2, \\
(n_2) \quad 0 &= (1+\lambda_2)u_n^2(c_2, n_2) + \gamma w_2 \mu_2.
\end{aligned}
$$

Dividing, we find

$$
\mathrm{MRS}^2(c_2, w_2 n_2) = 1.
$$

The marginal rate of substitution between consumption and labor is equalized with the wage, so we recover the "no distortion at the top" result: Type 2 agents must face a marginal tax rate of zero at their equilibrium income $w_2 n_2$. Similarly, type 1's first-order conditions are

$$
\begin{aligned}
(c_1) \quad 0 &= \eta u_c^1(c_1, n_1) - \lambda_2 u_c^2\left(c_1, \frac{w_1}{w_2} n_1\right) - \gamma\mu_1, \\
(n_1) \quad 0 &= \eta u_n^1(c_1, n_1) - \frac{w_1}{w_2}\lambda_2 u_n^2\left(c_1, \frac{w_1}{w_2} n_1\right) + \gamma w_1 \mu_1.
\end{aligned}
$$

Dividing yields

$$
\begin{aligned}
\mathrm{MRS}^1(c_1, w_1 n_1) &= \frac{1 - \left(\frac{\lambda_2}{w_2} u_n^2\left(c_1, \frac{w_1}{w_2} n_1\right)\right) / (\gamma\mu_1)}{1 + \left(\lambda_2 u_c^2\left(c_1, \frac{w_1}{w_2} n_1\right)\right) / (\gamma\mu_1)} \\
&= \mathrm{MRS}^2(c_1, w_1 n_1) + \frac{1 - \mathrm{MRS}^2(c_1, w_1 n_1)}{1+\nu},
\end{aligned}
$$

where

$$
\nu := \frac{\lambda_2 u_c^2\left(c_1, \frac{w_1}{w_2} n_1\right)}{\gamma\mu_1}.
$$

Rearranging the equation above and making use of the single-crossing assumption, we have

$$
\begin{aligned}
(1+\nu)\,\mathrm{MRS}^1(c_1, w_1 n_1) &= \nu \mathrm{MRS}^2(c_1, w_1 n_1) + 1 \\
&< \nu \mathrm{MRS}^1(c_1, w_1 n_1) + 1 \\
\Rightarrow \mathrm{MRS}^1(c_1, w_1 n_1) &< 1.
\end{aligned}
$$

Thus the marginal rate of substitution between consumption and labor is below the wage w_1, so type 1 agents face a positive marginal tax rate at their equilibrium income $w_1 n_1$.

3 Nonlinear Production

We now relax the assumption that the production function F is linear. In this case, when solving the Pareto efficiency problem we must be careful to incorporate the general equilibrium determination of the wages (w_1, w_2). The equilibrium relative wage is determined by

$$\frac{w_1}{w_2} = \frac{f(n) - nf'(n)}{f'(n)} =: \phi\left(\frac{n_2}{n_1}\right).$$

The incentive constraints can then be written

$$u^1(c_1, n_1) \geq u^1\left(c_2, \frac{n_2}{\phi(n_2/n_1)}\right),$$
$$u^2(c_2, n_2) \geq u^2\left(c_1, \phi\left(\frac{n_2}{n_1}\right) n_1\right).$$

Assuming $w_2 > w_1$ and $\lambda_2 > 0 = \lambda_1$ at the solution to the modified Pareto efficiency problem, we find the first-order conditions

$$
\begin{aligned}
(c_1) &\quad 0 = \eta u_c^1(c_1, n_1) - \lambda_2 u_c^2(c_1, n_1\phi) - \gamma\mu_1, \\
(n_1) &\quad 0 = \eta u_n^1(c_1, n_1) - \lambda_2 u_n^2(c_1, n_1\phi)\left(\phi - \frac{n_2}{n_1}\phi'\right) + \gamma F_1 \mu_1, \\
(c_2) &\quad 0 = (1 + \lambda_2) u_c^2(c_2, n_2) - \gamma\mu_2, \\
(n_2) &\quad 0 = (1 + \lambda_2) u_n^2(c_2, n_2) - \lambda_2 u_n^2(c_1, n_1\phi)\phi' + \gamma F_2 \mu_2.
\end{aligned}
$$

Dividing the first-order conditions for type 2, we find

$$\mathrm{MRS}^2(c_2, w_2 n_2) = 1 - \frac{\lambda_2 u_n^2(c_1, n_1\phi)\phi'}{\gamma w_2 \mu_2}.$$

Provided that $\phi' > 0$, we then find that the marginal rate of substitution between consumption and labor is strictly greater than the wage for type 2 agents, implying a *negative* marginal tax rate at the equilibrium income $w_2 n_2$. We can similarly use the first-order conditions for type 1 agents to show that they continue to face a positive marginal tax rate at the optimum, and that this marginal tax rate is decreasing in the elasticity of substitution between both types of labor in production.

The intuition for these results is as follows: When type 1 labor and type 2 labor are not perfectly substitutable in production, the relative quantities employed will affect the marginal product of each factor. By subsidizing employment of type 2 agents, the government raises the relative wage of type 1 agents, making it less attractive for a type 2 agent to mimick a type 1 agent. The government can then allow type 1 agents to work and consume more so as to increase their utilities while ensuring that the type 2 incentive constraint is relaxed relative to the fixed wage benchmark. In this sense, "predistribution" through manipulation of factor prices can benefit the government, an idea considered again in Naito (1999).

References

Naito, H. (1999). Re-examination of uniform commodity taxes under a non-linear income tax system and its implication for production efficiency. *Journal of Public Economics*, *71*(2), 165–188.

Stiglitz, J. E. (1982). Self-selection and pareto efficient taxation. *Journal of public economics*, *17*(2), 213–240.