Recitation 6: Nonlinear Taxation II

Todd Lensman

March 11, 2022

Recitation Plan: Review the heuristic derivation of the optimal nonlinear income tax formula in Saez (2001)

1 Model

Consumption. The economy has a unit measure of heterogeneous agents, where the density of agents of type $\theta \in \mathbb{R}_+$ is $f(\theta)$. An agent of type θ has preferences over consumption c and income (effective labor) y given by the utility function $u(c, y, \theta)$, assumed twice continuously differentiable, strictly concave, strictly increasing in c and θ, and strictly decreasing in y. We also assume that u satisfies the single-crossing condition, so that the marginal rate of substitution

$$-\frac{u_y(c, y, \theta)}{u_c(c, y, \theta)}$$

is decreasing in θ. Given an income tax schedule T, an additional linear tax τ, and an additional lump-sum grant I, a type θ agent solves

$$\max_{c,y} u(c, y, \theta) \quad \text{subject to} \quad c \leq y - T(y) - (1-\tau) y + I.$$

Let $y(\theta; T, \tau, I)$ denote the resulting income choice, which is weakly increasing in θ and strictly increasing wherever $y \mapsto T(y) - (1-\tau) y$ is smooth. Suppose that T is smooth, and define the uncompensated elasticity, the income effect, and the compensated elasticity as follows:

$$\begin{aligned}
\varepsilon^u(y; T) &:= \frac{1 - T'(y)}{y} \frac{\partial y(\theta; T, 0, 0)}{\partial (1-\tau)}, \\
\eta(y; T) &:= -\left(1 - T'(y)\right) \frac{\partial y(\theta; T, 0, 0)}{\partial I}, \\
\varepsilon^c(y; T) &:= \varepsilon^u(y; T) + \eta(y; T),
\end{aligned}$$

where $y = y(\theta; T, 0, 0)$. In the remainder of the note, I set $\tau = I = 0$ and write $y = y(\theta; T)$.

Production. The production technology is such that income is transformed one-for-one into consumption.

Government. The government uses the income tax schedule T to finance exogenous government expenditures E and to redistribute. The government's budget constraint is

$$E \leq \int_0^\infty T(y) H(dy; T),$$

where the endogenous income distribution H satisfies $H(y; T) := F\left(y^{-1}(\theta; T)\right)$. The government's redistributive objective is described by the welfare function

$$\int_0^\infty G(u(c(\theta), y(\theta))) f(\theta) d\theta.$$

Equilibrium. An equilibrium is a tuple $\left((c(\theta), y(\theta))_{\theta \in \mathbb{R}_+}, T\right)$ such that

(i) the government's budget constraint is satisfied;

(ii) each type θ agent chooses $(c(\theta), y(\theta))$ given the tax schedule T.

2 Heuristic Derivation of the Diamond-Saez Formula

The government's task is to choose a tax schedule T to maximize its welfare function, subject to its budget constraint. Let φ denote the multiplier on the resource constraint, which can be interpreted as the value of public funds at the optimum. To derive an optimality condition that characterizes the optimal tax schedule T, we use a variational argument: We consider small changes to the optimal tax schedule, and we argue (just as in finite-dimensional calculus) that as these changes become small, they must have no first-order effect on the government's Lagrangian (penalized objective function). In particular, for each income level y^*, we consider a variation that increases the marginal tax rate by $d\tau$ for all incomes between y^* and $y^* + dy^*$. The effect of this variation is to increase the marginal tax rate for all income levels $[y^*, y^* + dy^*]$ while increasing only the average tax rate for all income levels $[y^* + dy^*, \infty)$. The effect of this variation can be divided into three components.

Mechanical Effect. Holding behavioral responses constant, the variation has direct effects on (i) the government's budget constraint and (ii) the government's objective. In particular, every agent with income above y^*+dy^* pays additional income taxes $d\tau dy^*$. Any agent with income $y \in (y^* + dy^*)$ pays additional income taxes $d\tau\,(y-y^*)$. The total effect on the government's revenues is

$$d\tau \int_{y^*}^{y^*+dy^*} (y-y^*)h(y)\,dy + d\tau dy^* \int_{y^*+dy^*}^{\infty} h(y)\,dy,$$

where $h(y) := H'(y;T)$ is the income density at the optimal tax, which is assumed to exist. Note that the first term is of smaller order than the second term in $d\tau dy^*$: It tends to zero as $d\tau, dy^* \to 0$ when renormalized by $d\tau dy^*$, while the second term tends to a non-zero limit when renormalized by $d\tau dy^*$. We are only concerned about first-order changes around the optimal tax, so we can freely ignore the first term in our subsequent analysis. Similarly, the effect on the government's objective is

$$-d\tau dy^* \int_{y^*+dy^*}^{\infty} g(y)h(y)\,dy,$$

where I have ignored the second-order term and defined

$$g(y) := \frac{G'(u(c(\theta), y(\theta)))u_c(c(\theta), y(\theta))}{\varphi}$$

for $y = y(\theta;T)$. Combining the previous two terms, the first-order mechanical effect is

$$M := d\tau dy^* \int_{y^*}^{\infty} (1-g(y))h(y)\,dy.$$

Substitution Effect. We now consider the first of two behavioral responses: substitution effects for agents who initially selected incomes $y \in [y^* + dy^*]$. Any such agent now faces a higher marginal tax rate $d\tau$ and adjusts her income according to the compensated elasticity ε^c:

$$dy = -d\tau \frac{y}{1-T'(y)}\varepsilon^c(y).$$

By the Envelope Theorem, this behavioral adjustment has no first-order effect on the agent's utility, so it does not impact the government's objective. However, it does have a first-order effect (“fiscal externality”) on the government’s budget constraint:

$$\int_{y^*}^{y^*+dy^*} T'(y)\left[-d\tau\frac{y}{1-T'(y)}\varepsilon^c(y)\right]h(y)\,dy.$$

For dy^* small, the total effect of substitution on the budget constraint is

$$S := -d\tau dy^*\frac{y^*T'(y^*)}{1-T'(y^*)}\varepsilon^c(y^*)h(y^*).$$

Income Effect. The second behavioral response results from income effects for agents who initially selected incomes $y \in [y^* + dy^*, \infty)$. Any such agent faces the same marginal tax rate $T'(y)$, but with her after-tax income reduced by $d\tau dy^*$. She then adjusts her income according to the income effect η:

$$dy = -d\tau dy^*\frac{\eta(y)}{1-T'(y)}.$$

Again, this adjust has first-order effect only on the government’s budget constraint:

$$I := -d\tau dy^*\int_{y^*}^{\infty}\frac{T'(y)}{1-T'(y)}\eta(y)h(y)\,dy.$$

Optimality. At an optimal tax schedule T, the mechanical, substitution, and income effects must sum to zero:

$$0 = M + S + I.$$

Replacing $y = y^*$, this equation can be written

$$\frac{T'(y)}{1-T'(y)}\varepsilon^c(y)yh(y) = \int_y^{\infty}(1-g(\tilde{y}))h(\tilde{y})\,d\tilde{y} - \int_y^{\infty}\frac{T'(\tilde{y})}{1-T'(\tilde{y})}\eta(\tilde{y})h(\tilde{y})\,d\tilde{y}. \qquad (1)$$

Remarkably, we can differentiate this equation to derive the Pareto optimality inequality of Werning (2007):

Proposition 2.1 (Werning, 2007)**.** A tax schedule T is optimal for some income-dependent

Pareto weights $g(z) \geq 0$ only if

$$-\frac{T'(y)}{1-T'(y)}\varepsilon^c(y)\left[\frac{\partial\log(yh(y))}{\partial\log(y)}+\frac{\partial\log\left(\frac{T'(y)}{1-T'(y)}\varepsilon^c(y)\right)}{\partial\log(y)}+\frac{\eta(y)}{\varepsilon^c(y)}\right]\leq 1.$$

Alternatively, we can view (1) as an ordinariy differential equation with variable coefficients:

$$\underbrace{\frac{T'(y)}{1-T'(y)}\eta(y)h(y)}_{\dot{K}(y)}=\underbrace{\frac{\eta(y)}{\varepsilon^c(y)y}}_{D(y)}\left[\underbrace{\int_y^\infty(1-g(\tilde{y}))h(\tilde{y})\,d\tilde{y}}_{C(y)}-\underbrace{\int_y^\infty\frac{T'(\tilde{y})}{1-T'(\tilde{y})}\eta(\tilde{y})h(\tilde{y})\,d\tilde{y}}_{K(y)}\right].$$

Integrating with the boundary condition $\lim_{y\to\infty}K(y)=0$ yields

$$\begin{aligned}K(y)&=-\int_y^\infty D(\tilde{y})C(\tilde{y})\exp\left(-\int_y^{\tilde{y}}D(z)\,dz\right)d\tilde{y}\\&=-\int_y^\infty C'(\tilde{y})\exp\left(-\int_y^{\tilde{y}}D(z)\,dz\right)d\tilde{y}-C(y),\end{aligned}$$

where the second line follows by integrating by parts. A final differentiation gives the standard Diamond-Saez formula:

$$\frac{T'(y)}{1-T'(y)}=\frac{1-H(y)}{\varepsilon^c(y)yh(y)}\int_y^\infty(1-g(\tilde{y}))\exp\left(\int_y^{\tilde{y}}\frac{\eta(z)}{\varepsilon^c(z)}\frac{dz}{z}\right)\frac{h(\tilde{y})\,d\tilde{y}}{1-H(y)}. \tag{2}$$

References

Saez, E. (2001). Using elasticities to derive optimal income tax rates. *The review of economic studies*, *68*(1), 205–229.

Werning, I. (2007). *Pareto efficient income taxation*. mimeo, MIT.