Mixed Taxation and the Atkinson-Stiglitz Theorem

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These notes describe a standard model of *mixed taxation* in which a government chooses a nonlinear income tax and linear commodity taxes to raise revenue and redistribute between heterogeneous agents (e.g., Atkinson and Stiglitz, 1976; Mirrlees, 1986). I provide two proofs of the Atkinson-Stiglitz Theorem, which describes conditions under which Pareto optimal tax policies feature uniform taxation of all commodities (other than labor income). The first proof formalizes the argument of Atkinson and Stiglitz (1976), relying on techniques from optimal control to derive necessary conditions for optimal income and commodity taxes. The second proof follows Iván Werning's generalization of Laroque's (2005) argument, which provides an algorithm to construct a Pareto-improving reform to any tax system with differential commodity taxation.

1 Model

The economy is static and consists of a unit measure of agents and a finite number of representative firms. Firms are indexed by $i \in \{1, \ldots, N\}$ and produce differentiated *final goods* using heterogeneous production functions $f_i : \mathbb{R}_+ \to \mathbb{R}_+$ that map *effective labor* $y \in \mathbb{R}_+$ to output. I assume that each f_i is strictly increasing, weakly concave, and twice differentiable on $\mathbb{R}_{++}$, with $f_i(0) = 0$. Firms behave competitively in factor and output markets. The wage is normalized to one, and given an output price p_i, firm i solves

$$\pi_i := \max_{y \geq 0} p_i f_i(y) - y. \tag{1}$$

Let y_f^i denote the resulting effective labor demand.

Agents receive idiosyncratic *types* $\theta \in \left[\underline{\theta}, \bar{\theta}\right] \subset \mathbb{R}_{++}$ distributed according to a density function g. An agent with type θ consumes a quantity $c_i \in \mathbb{R}_+$ of final good i at price $(1 + t_i)p_i$ and supplies effective labor $y \in [0, \theta)$ at unit wage. The agent's preferences over consumption and effective labor are represented by a weakly separable utility function $U(\alpha(c), y, \theta)$, where

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$c := (c_i)_{i=1}^N$. I assume that U satisfies the following technical conditions:

(i) U is twice continuously differentiable on $\mathbb{R} \times (0, \theta) \times (\underline{\theta}, \bar{\theta})$, with $U_\alpha, -U_y > 0$;

(ii) U is weakly concave in (α, y);

(iii) $\lim_{y \uparrow \theta} U(\alpha, y, \theta) = -\infty \ \forall \alpha, \theta$;

(iv) (Spence-Mirrlees) $\theta \mapsto -U_y(\alpha, y, \theta) / U_\alpha(\alpha, y, \theta)$ is strictly decreasing $\forall \alpha, y$.

Similarly, the aggregator $\alpha : \mathbb{R}_+^N \to \mathbb{R}$ satisfies the following:

(i) α is twice continuously differentiable on $\mathbb{R}_{++}^N$, with $\alpha_{c_i} > 0$;

(ii) α is weakly concave;

(iii) $\lim_{c_i \downarrow 0} \alpha_{c_i}(c_i, c_{-i}) = \infty \ \forall c_{-i} := (c_j)_{j \neq i}$.

Given prices p, linear commodity taxes $t = (t_i)_{i=1}^n$, and a potentially nonlinear income tax T, an agent with type θ solves the consumer choice problem

$$V^\theta := \max_{y \in [0,\theta), c_i \geq 0} U(\alpha(c), y, \theta) \quad \text{subject to} \quad \sum_{i=1}^N (1 + t_i) p_i c_i \leq y - T(y). \tag{2}$$

I restrict to differentiable taxes T to ensure the existence of a solution $\left\{y_a^\theta, c_i^\theta\right\}$ to the agent's problem (2) and to facilitate the derivation of optimality conditions.

The economy also has a government that seeks to redistribute among agents while financing exogenous purchases of final goods $G_i \in \mathbb{R}_+$, $G := (G_i)_{i=1}^N$. To do so, it chooses the commodity taxes t and the income tax T, subject to a resource constraint. I also assume initially that firm profits are fully taxed. The government then solves the problem

$$\sup_{t,T} \int V^\theta \rho(\theta) g(\theta) d\theta \tag{3}$$

subject to

$$\int \left[T\left(y_a^\theta\right) + \sum_{i=1}^N t_i p_i c_i^\theta \right] g(\theta) d\theta + \sum_{i=1}^N \pi_i \geq \sum_{i=1}^N p_i G_i, \tag{4}$$

$$y \geq T(y) \quad \forall y \geq 0. \tag{5}$$

Here $\left\{y_a^\theta, c_i^\theta, y_f^i, p\right\}$ consitutes an equilibrium of the economy given t, T, defined as follows:

Definition 1. Given commodity taxes t and an income tax T, an *equilibrium* is an agent allocation $\left\{y_a^\theta, c_i^\theta\right\}$, a firm allocation $\left\{y_f^i\right\}_{i=1}^n$, and prices p such that

(i) an agent with type θ chooses $\left\{y_a^\theta, c_i^\theta\right\}$ to solve (2) given t, T, and p;

(ii) firm i chooses y_f^i to solve (1) given p_i;

(iii) the labor market clears:

$$0 = \int y_a^\theta g(\theta)\, d\theta - \sum_{i=1}^N y_f^i; \tag{6}$$

(iv) the goods markets clear:

$$0 = \int c_i^\theta g(\theta)\, d\theta + G_i - f_i\left(y_f^i\right) \quad i = 1, \ldots, N. \tag{7}$$

Note that I have assumed away public production for simplicity.

2 Uniform Commodity as a Necessary Condition

In this section, I reformulate the government's problem (3) as an optimal control problem, use Pontryagin's Maximum Principle to derive optimality conditions, and provide a restriction on preferences under which the optimality conditions require uniform taxation of commodities.

2.1 Implementable Equilibria

First consider the agent's problem (2), and let $q_i := (1 + t_i)\, p_i$ denote the consumer-facing price for good i. The optimization problem can be separated into two stages as follows:

$$V^\theta = \max_{y \in [0,\theta)} U\left(\tilde{\alpha}(q, y - T(y)), y, \theta\right), \tag{8}$$

where

$$\tilde{\alpha}(q, R) := \max_{c_i \geq 0} \alpha(c) \quad \text{subject to} \quad q \cdot c \leq R. \tag{9}$$

Since α is weakly concave and satisfies Inada conditions at zero, the following interior first order conditions are necessary and sufficient to characterize the solution $c(p,R)$ to (9) for $p >> 0$ and $R > 0$:

$$\alpha_{c_i}(c(q,R)) - q_i \lambda(q,R) = 0 \quad \forall i, \tag{10}$$

$$R - \sum_{i=1}^{N} q_i c_i(q,R) = 0. \tag{11}$$

Here $\lambda(q,R) > 0$ is the Lagrange multiplier associated with the binding budget constraint.

Now consider the firm's problem (1). Assuming an interior solution, the following first order condition is necessary and sufficient to characterize y_f^i:

$$p_i = \frac{1}{f_i'\left(y_f^i\right)} \tag{12}$$

With the above calculations, the following implementation result holds:

Lemma 1. Consider a labor supply function y_a^θ, a retained income function R^θ, consumer prices q, and labor demands y_f^i such that

(i) y_a^θ, R^θ are absolutely continuous, y_a is non-decreasing, and $V^\theta := U\left(\tilde{\alpha}\left(q,R^\theta\right), y_a^\theta, \theta\right)$ satisfies the evolution equation

$$\dot{V}^\theta = U_\theta\left(\tilde{\alpha}\left(q,R^\theta\right), y_a^\theta, \theta\right); \tag{13}$$

(ii) $y_f^i > 0$ for $i = 1, \ldots, N$;

(iii) the market-clearing conditions (6, 7) are satisfied.

Then there exist commodity taxes t and an income tax T such that $\left\{y_a^\theta, c\left(q,R^\theta\right), y_f^i, p\right\}$ constitute an equilibrium given t, T, where p_i is defined by the firm first order condition (12).

Proof: The market-clearing conditions (6, 7) hold by assumption, and firm optimality is ensured because prices p are defined to satisfy the firm's interior first order condition (12), which is necessary and sufficient since the firm's problem (1) is convex.

Define the commodity taxes t such that $q_i = (1+t_i)p_i$, and define the income tax T on the image of y_a by $T\left(y_a^\theta\right) := y_a^\theta - R^\theta$. To show that T is well-defined, I first show that (13) implies that the following incentive-compatibility conditions are satisfied:

$$V^{\theta} \geq U\left(\tilde{\alpha}\left(q, R^{\hat{\theta}}\right), y_a^{\hat{\theta}}, \theta\right) \quad \forall \theta, \hat{\theta}. \tag{14}$$

To see this, note that since y_a^{θ}, R^{θ} are absolutely continuous, the Fundamental Theorem of Calculus implies

$$\begin{aligned}
&V^{\theta} - U\left(\tilde{\alpha}\left(q, R^{\hat{\theta}}\right), y_a^{\hat{\theta}}, \theta\right) \\
&= \int_{\hat{\theta}}^{\theta} U_{\alpha}\left(\tilde{\alpha}\left(q, R^s\right), y_a^s, \theta\right) \tilde{\alpha}_R\left(q, R^s\right) \dot{R}^s - U_y\left(\tilde{\alpha}\left(q, R^s\right), y_a^s, \theta\right) \dot{y}_a^s ds \\
&= \int_{\hat{\theta}}^{\theta} U_{\alpha}\left(\tilde{\alpha}\left(q, R^s\right), y_a^s, \theta\right) \left[\tilde{\alpha}_R\left(q, R^s\right) \dot{R}^s - \frac{U_y\left(\tilde{\alpha}\left(q, R^s\right), y_a^s, \theta\right)}{U_{\alpha}\left(\tilde{\alpha}\left(q, R^s\right), y_a^s, \theta\right)} \dot{y}_a^s\right] ds \\
&\geq \int_{\hat{\theta}}^{\theta} U_{\alpha}\left(\tilde{\alpha}\left(q, R^s\right), y_a^s, \theta\right) \left[\tilde{\alpha}_R\left(q, R^s\right) \dot{R}^s - \frac{U_y\left(\tilde{\alpha}\left(q, R^s\right), y_a^s, s\right)}{U_{\alpha}\left(\tilde{\alpha}\left(q, R^s\right), y_a^s, s\right)} \dot{y}_a^s\right] ds \\
&= 0.
\end{aligned}$$

The inequality follows from the Spence-Mirrlees condition and the assumption that y_a is non-decreasing. The final line holds because the envelope condition (13) is equivalent to

$$0 = U_{\alpha}\left(\tilde{\alpha}\left(q, R^{\theta}\right), y_a^{\theta}, \theta\right) \tilde{\alpha}_R\left(q, R^{\theta}\right) \dot{R}^{\theta} - U_y\left(\tilde{\alpha}\left(q, R^{\theta}\right), y_a^{\theta}, \theta\right) \dot{y}_a^{\theta} \quad \forall \theta.$$

Given that the incentive-compatibility conditions (14) are satisfied and that $\tilde{\alpha}$ is strictly increasing in R, there cannot exist θ, θ' such that $y_a^{\theta} = y_a^{\theta'}$ and $R^{\theta} \neq R^{\theta'}$. Hence T is well-defined on $\left[y_a^{\underline{\theta}}, y_a^{\bar{\theta}}\right]$, and we can set T arbitrarily high on the complement of this set to ensure that such labor supply choices are not optimal for any agent. It follows immediately from the two-stage decomposition (8, 9) that $\left(c\left(q, R^{\theta}\right), y_a^{\theta}\right)$ solves (2) given t, T, and p. $\square$

For a final simplification, note that Walras's Law implies that the labor market clearing constraint (6) is satisfied if the government's resource constraint (4) and all goods market clearing constraints (7) are satisfied. Using the firm first order condition (12) to define firm prices p, the government's resource constraint (4) can be written

$$\begin{aligned}
0 &\leq \int \left\{ y_a^{\theta} - R^{\theta} + \sum_{i=1}^{N} \left[\left(q_i - \frac{1}{f_i'\left(y_f^i\right)} \right) c_i^{\theta} + \frac{f_i\left(y_f^i\right)}{f_i'\left(y_f^i\right)} - y_f^i - \frac{G_i}{f_i'\left(y_f^i\right)} \right] \right\} g(\theta) d\theta \\
&= \int \left\{ y_a^{\theta} - R^{\theta} + \sum_{i=1}^{N} \left[q_i c_i^{\theta} - y_f^i \right] \right\} g(\theta) d\theta.
\end{aligned}$$

The second line follows from the goods market clearing constraints (7).

2.2 Optimality Conditions

Given the implementation result of Lemma 1, the government's problem (3) can be formulated as an optimal control problem as follows:

$$\sup_{V, y_a, y_f, q} \int V^{\theta} \rho(\theta) g(\theta) d\theta \tag{15}$$

subject to

$$\int \left\{ y_a^{\theta} - R\left(V^{\theta}, y_a^{\theta}, q, \theta\right) + \sum_{i=1}^{N} \left[q_i c_i \left(q, R\left(V^{\theta}, y_a^{\theta}, q, \theta\right)\right) - y_f^i \right] \right\} g(\theta) d\theta \geq 0, \tag{16}$$

$$\int \left\{ c_i \left(q, R\left(V^{\theta}, y_a^{\theta}, q, \theta\right)\right) + G_i - f_i\left(y_f^i\right) \right\} g(\theta) d\theta = 0 \quad i = 1, \ldots, N, \tag{17}$$

$$\dot{V}^{\theta} = U_{\theta}\left(\tilde{\alpha}\left(q, R\left(V^{\theta}, y_a^{\theta}, q, \theta\right)\right), y_a^{\theta}, \theta\right), \tag{18}$$

$$\dot{y}_f^i = 0 \quad i = 1, \ldots, N, \tag{19}$$

$$\dot{q}_i = 0 \quad i = 1, \ldots, N, \tag{20}$$

$$R\left(V^{\theta}, y_a^{\theta}, q, \theta\right) \geq 0 \quad \forall \theta. \tag{21}$$

The implicit retained income function R is defined by

$$V = U\left(\tilde{\alpha}\left(q, R(V, y, q, \theta)\right), y, \theta\right). \tag{22}$$

In addition, I adopt the normalization

$$q_1 = p_1 = \frac{1}{f_1'\left(y_f^1\right)}. \tag{23}$$

To derive optimality conditions for (15), note that the non-negativity constraint (21) is non-binding since U satisfies Inada conditions at zero in the consumption goods c_i. Let $\mu_1 > 0$ denote the static multiplier on the resource constraint (16), let μ_2^i denote the static multiplier on the market-clearing constraint (17), let φ_V denote the costate for V, let $\varphi_{y_f^i}$ denote the costate for y_f^i, and let φ_{q_i} denote the costate for q_i. Suppressing its dependence on its arguments for notational simplicity, define the Hamiltonian

$$\begin{aligned} H &:= V\rho(\theta)g(\theta) \\ &+ \mu_1\left\{y_a - R(V,y_a,q,\theta) + \sum_{i=1}^{N}\left[q_i c_i(q,R(V,y_a,q,\theta)) - y_f^i\right]\right\}g(\theta) \\ &+ \sum_{i=1}^{N}\mu_2^i\left[c_i(q,R(V,y_a,q,\theta)) + G_i - f_i\left(y_f^i\right)\right]g(\theta) \\ &+ \varphi_V U_\theta\left(\tilde{\alpha}(q,R(V,y_a,q,\theta)), y_a, \theta\right). \end{aligned} \tag{24}$$

Pontryagin's Maximum Principle then implies the existence of multipliers $\mu_1 > 0$, μ_2^i and costates φ_V, $\varphi_{y_f^i}$, φ_{q_i} such that

(i) V, y_f^i, q_i follow the evolution equations

$$\dot{V}^\theta = H_{\varphi_V}^\theta = U_\theta\left(\tilde{\alpha}\left(q,R\left(V^\theta,y_a^\theta,q,\theta\right)\right),y_a^\theta,\theta\right), \tag{25}$$

$$\dot{y}_f^i = H_{\varphi_{y_f^i}}^\theta = 0 \quad i=1,\ldots,N, \tag{26}$$

$$\dot{q}_i = H_{\varphi_{q_i}}^\theta = 0 \quad i=1,\ldots,N; \tag{27}$$

(ii) φ_V, $\varphi_{y_f^j}$, φ_{q_j} follow the evolution equations

$$\begin{aligned} \dot{\varphi}_V^\theta &= -H_V^\theta \\ &= \left\{-\rho(\theta) + \mu_1 R_V^\theta - R_V^\theta\sum_{i=1}^{N}\left(\mu_1 q_i + \mu_2^i\right)\frac{\partial c_i^\theta}{\partial R}\right\}g(\theta) - \varphi_V^\theta U_{\alpha\theta}\tilde{\alpha}_R^\theta R_V^\theta \\ &= \left\{-\rho(\theta) + \frac{\mu_1}{U_\alpha^\theta\tilde{\alpha}_R^\theta} - \frac{1}{U_\alpha^\theta\tilde{\alpha}_R^\theta}\sum_{i=1}^{N}\left(\mu_1 q_i + \mu_2^i\right)\frac{\partial c_i^\theta}{\partial R}\right\}g(\theta) - \varphi_V^\theta\frac{U_{\alpha\theta}^\theta}{U_\alpha^\theta}, \end{aligned} \tag{28}$$

$$\begin{aligned} \dot{\varphi}_{y_f^j} &= -H_{y_f^j}^\theta \\ &= \mu_1 g(\theta) + \mu_2^j f_j'\left(y_f^j\right)g(\theta) \quad j=1,\ldots,N, \end{aligned} \tag{29}$$

$$\begin{aligned} \dot{\varphi}_{q_j}^\theta &= -H_{q_j}^\theta \\ &= \left\{\mu_1\left(R_{q_j}^\theta - c_j^\theta\right) - \sum_{i=1}^{N}\left(\mu_1 q_i + \mu_2^i\right)\left(\frac{\partial c_i^\theta}{\partial q_j} + \frac{\partial c_i^\theta}{\partial R}R_{q_j}^\theta\right)\right\}g(\theta) \\ &\quad - \varphi_V^\theta U_{\alpha\theta}^\theta\left(\tilde{\alpha}_{q_j}^\theta + \tilde{\alpha}_R^\theta R_{q_j}^\theta\right) \end{aligned} \tag{30}$$

$$= -\sum_{i=1}^{N}\left(\mu_1 q_i + \mu_2^i\right)\left(\frac{\partial c_i^\theta}{\partial q_j} + \frac{\partial c_i^\theta}{\partial R}c_j^\theta\right)g(\theta) \quad j=2,\ldots,N; \tag{31}$$

(iii) φ_V, $\varphi_{y_f^j}$, φ_{q_j} satisfy the transversality conditions

$$0 = \varphi_V^{\underline{\theta}} = \varphi_V^{\bar{\theta}} = \varphi_{y_f^j}^{\underline{\theta}} = \varphi_{y_f^j}^{\bar{\theta}} = \varphi_{q_j}^{\underline{\theta}} = \varphi_{q_j}^{\bar{\theta}}; \tag{32}$$

(iv) y_a maximizes the Hamiltonian pointwise:

$$y_a^\theta \in \arg\max_{y \in [0,\theta)} H(\cdot,\theta) \quad \forall \theta. \tag{33}$$

For any θ such that $y_a^\theta > 0$, the Hamiltonian maximization condition (33) yields the first order condition

$$0 = \mu_1 \left(1 - R_{y_a}^\theta\right) g(\theta) + g(\theta) \sum_{i=1}^{N} \left(\mu_1 q_i + \mu_2^i\right) \frac{\partial c_i^\theta}{\partial R} R_{y_a}^\theta + \varphi_V^\theta \left(U_{\alpha\theta}^\theta \tilde{\alpha}_R^\theta R_{y_a}^\theta + U_{y\theta}^\theta\right) \tag{34}$$

$$= \mu_1 \left(1 + \frac{U_y^\theta}{U_\alpha^\theta \tilde{\alpha}_R^\theta}\right) g(\theta) - \frac{U_y^\theta}{U_\alpha^\theta \tilde{\alpha}_R^\theta} g(\theta) \sum_{i=1}^{N} \left(\mu_1 q_i + \mu_2^i\right) \frac{\partial c_i^\theta}{\partial R} + \varphi_V^\theta \left(U_{y\theta}^\theta - U_{\alpha\theta}^\theta \frac{U_y^\theta}{U_\alpha^\theta}\right) \tag{35}$$

2.3 Proof of Uniform Commodity Taxation

We can determine the structure of optimal commodity taxes by examining the optimality conditions with respect to y_f^j and q_j, i.e., the integral equations satisfied by the costates $\varphi_{y_f^j}$ and φ_{q_j}. First note that the $\varphi_{y_f^j}$ evolution equations (29) and the corresponding transversality conditions (32) imply

$$\mu_2^j = -\frac{\mu_1}{f_j'\left(y_f^j\right)} = -p_j \mu_1 \quad j = 1, \ldots, N. \tag{36}$$

Integrating the φ_{q_j} evolution equations (31) and substituting then give

$$0 = \sum_{i=1}^{N} \left(q_i - \frac{1}{f'\left(y_f^i\right)}\right) \int \left(\frac{\partial c_i^\theta}{\partial q_j} + \frac{\partial c_i^\theta}{\partial R} c_j^\theta\right) g(\theta)\, d\theta \tag{37}$$

$$\Rightarrow 0 = \sum_{i=2}^{N} t_i p_i \int \left(\frac{\partial c_i^\theta}{\partial q_j} + \frac{\partial c_i^\theta}{\partial R} c_j^\theta\right) g(\theta)\, d\theta \quad j = 2, \ldots, N. \tag{38}$$

This final system of equations can be written in matrix form:

$$\left[\mathbb{E}_g S^\theta\right] (t_2 p_2, \ldots, t_N p_N)^\top = 0. \tag{39}$$

Here S^{θ} is the type-θ agent's substitution matrix for goods $i = 2\ldots,N$. Assuming that this matrix is negative definite (equivalently, that the expenditure function for goods $i = 2\ldots,N$ is *strictly* concave in q_{-1}), the cross-sectional average $\mathbb{E}_g S^{\theta}$ is negative definite. As a result, the unique solution to the system (39) is $t_{-1} = 0$, and we recover uniform commodity taxation as a necessary implication of optimality in the government's problem (3).

Theorem 1 (Atkinson & Stiglitz 1976)**.** If the expenditure function corresponding to (9) is strictly concave in q_{-1}, then any solution to (3) features $t_1 = \ldots = t_N$.

With this result, the remaining conditions that characterize the optimal income tax schedule T simplify substantially. In particular, the evolution equation (28) for φ_V and the Hamiltonian maximization first order condition (35) are respectively

$$\dot{\varphi}_V^{\theta} = \left(\frac{\mu_1}{U_\alpha^\theta \tilde{\alpha}_R^\theta} - \rho(\theta)\right) g(\theta) - \varphi_V^\theta \frac{U_{\alpha\theta}^\theta}{U_\alpha^\theta}, \tag{40}$$

$$0 = \mu_1 \left(1 + \frac{U_y^\theta}{U_\alpha^\theta \tilde{\alpha}_R^\theta}\right) g(\theta) + \varphi_V^\theta \left(U_{y\theta}^\theta - U_{\alpha\theta}^\theta \frac{U_y^\theta}{U_\alpha^\theta}\right) \quad \forall \theta : y_a^\theta > 0. \tag{41}$$

These are precisely the optimality conditions that obtain in the one-good Mirrlees model with linear production, so the theory of optimal income taxation is unchanged with the addition of multiple goods and potentially decreasing returns to scale under the assumptions of the Atkinson-Stiglitz Theorem.

3 Uniform Commodity Taxation without Loss of Generality

Theorem 1 provides conditions under which an optimal tax schedule must feature uniform commodity taxation, or equivalently no taxation of commodities at all. Although informative about the structure of optimal mixed taxation, a necessity theorem is not quite the result we would like when forming intuition about how to design real income and commodity taxes. Rather, a theorem describing conditions under which a nonlinear income tax is *sufficient* for the government to raise revenue and redistribute would provide substantive insight into the roles played by different policy instruments and allow us to understand if commodity taxes are useful in cases not covered by the hypotheses of Theorem 1.[1] In addition, the proof of the necessity result required a lengthy derivation of optimality conditions, which was only tractable because we restricted the policy instruments available to the government to a nonlinear income tax and linear commodity taxes. Though reasonable, we may ask if the result breaks when

[1]For example, is it really crucial for the agents' expenditure function to be *strictly* concave in prices?

considering slightly more general policy instruments, e.g., a potentially nonlinear tax of the form $T(y,c)$.

In this section, I prove Iván Werning's generalization of Laroque's (2005) sufficiency theorem describing conditions under which the government can raise revenue and redistribute optimally without taxing commodities. The proof provides greater intuition than that of Theorem 1 for uniform commodity taxation and highlights the importance of two key assumptions: weak separability of the utility function and homogeneity of the aggregator α. However, it does not apply immediately to the case in which the production functions f_i display variable marginal costs.

To motivate, consider the model described above, but with two modifications: First, suppose that the production functions f_i display constant returns to scale, i.e., $f_i(y) = y/\kappa_i$ for $\kappa_i \in \mathbb{R}_{++}$. Under this assumption, producer prices p are exogenously fixed at $p_i = \kappa_i$, and firms make zero profits in equilibrium. Second, suppose that the government can implement an arbitrary but anonymous budget set $B \subseteq \mathbb{R}_+^{N+1}$ for the agents. The consumer choice problem (2) then becomes

$$V^\theta(B) := \max_{y\in[0,\theta),c_i\geq 0} U(\alpha(c),y,\theta) \quad \text{subject to} \quad (c,y)\in B. \tag{42}$$

I restrict the government to budget sets B such that a solution to this problem exists $\forall\theta$. The corresponding resource constraint for the government is

$$\int \left[y_a^\theta - p\cdot c^\theta\right] g(\theta)\, d\theta \geq p\cdot G. \tag{43}$$

The argument will consist of showing that for any budget set B that implements an equilibrium, we can construct another budget set $\tilde{B}$ of the form

$$\tilde{B} = \left\{(c,y)\in\mathbb{R}_+^{N+1} \,\middle|\, p\cdot c \leq R(y)\right\} \tag{44}$$

such that each agent attains weakly higher utility and the government satisfies its resource constraint with equality under $\tilde{B}$. The intuition is as follows: By weak separability, each agent's utility depends only on her total labor supply y_a^θ and her final level of the consumption aggregate $\alpha^\theta := \alpha\left(c^\theta\right)$. As a result, each agent is weakly better off when her budget set is modified such that $\left(\alpha^\theta, y_a^\theta\right)$ remains feasible. For example, we could tax income y_a^θ such that her after-tax income is just sufficient to achieve consumption aggregate level α^θ when she faces competitive prices p. Without homogeneous preferences over consumption goods, we may not be able to make this modification using a type-independent tax schedule: An issue arises if two types choose the same labor supply under B, but one type requires strictly greater after-tax income to achieve the same level of her consumption aggregate at competitive prices. Preference homogeneity rules out this situation.

For the rigorous argument, let $\alpha^\theta := \alpha\left(c^\theta\right)$ denote the level of the consumption aggregate attained by an agent when solving (42), and let $e(p,\alpha)$ denote the expenditure function for the inner optimization problem (9), i.e.,

$$e(p,\alpha) := \min_{c_i \geq 0} p \cdot c \quad \text{subject to} \quad \alpha(c) \geq \alpha. \tag{45}$$

Define the retention function $R : \mathbb{R}_+ \to \mathbb{R}$ such that $R\left(y^\theta\right) := e\left(p,\alpha^\theta\right)\ \forall\theta$, and $R(y) < 0$ otherwise. Since α is continuous and strictly increasing,

$$V^\theta\left(\tilde{B}\right) = \max_{\theta'} U\left(\alpha^{\theta'}, y^{\theta'}, \theta\right) = V^\theta(B).$$

Thus all agents maintain the same labor supply y_a^θ and attain the same level of the consumption aggregate α^θ under $\tilde{B}$. However, the new consumption vector $\tilde{c}^\theta$ may be different from the previous consumption vector c^θ since the agents now allocate their after-tax income based on a standard Walrasian budget constraint with competitive prices. In any case, by the definition of the retention function R, the market value of the new consumption vector must be weakly lower than that of the old consumption vector:

$$p \cdot \tilde{c}^\theta = e\left(p,\alpha^\theta\right) \leq p \cdot c^\theta.$$

Net revenues for the government under $\tilde{B}$ must then satisfy

$$\int \left[y_a^\theta - p \cdot \tilde{c}^\theta\right] g(\theta)\, d\theta \geq \int \left[y_a^\theta - p \cdot c^\theta\right] g(\theta)\, d\theta.$$

In particular, the resource constraint (43) must be satisfied using $\tilde{B}$. Assuming that y_a^θ and $\tilde{c}^\theta$ are continuous with respect to lump-sum transfers, the inequality can be reduced to an equality by shifting R by a (non-negative) additive constant. This change must weakly raise each agent's utility, and with constant marginal costs, it is immediate that the resulting total labor supply can be allocated across firms to satisfy market-clearing constraints. We can then conclude that tax systems with uniform commodity taxes are sufficient for optimal mixed taxation under constant marginal costs:

Theorem 2 (Laroque 2005)**.** If the production functions f_i display constant marginal costs, then for any budget set B such that a solution to (42) exists $\forall \theta$, there exists an income tax $T : \mathbb{R}_+ \to \mathbb{R}$ such that $(0, T)$ implements an equilibrium that Pareto dominates the equilibrium corresponding to B.

The proof of this result does not generalize immediately when the production functions display variable marginal costs because prices are then endogenous to the tax system. In this case, Theorem 1 demonstrates that uniform commodity taxation is necessary for optimal mixed taxation when production functions display non-decreasing marginal costs and when preferences satisfy an additional mild concavity assumption.

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